



Independent protection in graphs



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ABSTRACT

Mobile guards on the vertices in an independent set X of a graph G are used to defend G against an infinite sequence of attacks on up to k independent vertices at a time. If such a set X exists, then G is said to be k -i-protectable. The k -independent protection number $i_k^\infty(G)$ is the smallest number of guards required to protect G against an arbitrary sequence of such simultaneous attacks.

We consider the problem of determining classes of k -i-protectable graphs. We introduce a colouring scheme called multicolourings as aid to determine whether a graph is k -i-protectable, and to describe a protection strategy if it is.

We show that any well-covered graph (i.e., all maximal independent sets have the same cardinality k) is k -i-protectable, even cycles in general are 1-i-protectable but not 2-i-protectable, and odd cycles in general are 2-i-protectable but not 3-i-protectable. We characterize k -i-protectable trees and give methods to construct larger k -i-protectable graphs from smaller ones.

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1. Introduction

Graph protection involves the deployment of defense units, such as guards, surveillance or broadcasting equipment, emergency services, etc., at strategic locations in a network, that is, at the vertices of a graph, two vertices being adjacent if the corresponding positions in the network are, in some appropriate sense, reachable from each other. The defense units move across edges of the graph to other vertices to deal with an event, such as an attack, a burglary, a fire, a medical or other emergency; every vertex of the graph must have access to a defense unit in this way.

A variety of protection models have been studied. Cases where defenders return to their original positions after dealing with an event before facing subsequent events include Roman domination, first studied in its modern reincarnation in [7], weak Roman domination [15] and secure domination [4].

We focus on securing the vertices of graphs against infinite sequences of events, henceforth called *attacks*, by stationing defense units, henceforth called *guards*, at the vertices. At most one guard is stationed at each vertex, and guards that move in response to an attack do not return to their original positions before facing another attack. We refer to such models as *eternal*, as they can be thought of as protecting a graph for eternity. A number of different eternal protection models have been studied.

For a finite, undirected graph $G = (V, E)$, let $\{D_i\}$, $D_i \subseteq V$, $i \geq 1$, be a collection of sets of vertices of the same cardinality, with initially one guard located on each vertex of D_1 . Each protection strategy can be modelled as a two-player game between a *defender* and an *attacker*: the defender chooses D_1 as well as each D_i , $i > 1$, while the attacker chooses the locations of the

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attacks $r_1, r_2, \dots$. Each attack is dealt with by the defender by choosing the next D_i subject to some constraints that depend on the particular game. The defender wins the game if they can successfully defend any sequence of attacks, subject to the constraints of the game described below; the attacker wins otherwise.

A vertex is *protected* if there is a guard on the vertex or on an adjacent vertex. A vertex v is *occupied* if there is a guard on v , otherwise v is *unoccupied*. An attack is *defended* if a guard moves to the attacked vertex.

For the **eternal domination problem**, each $D_i, i \geq 1$, is a dominating set, $r_i \in V$ (assume without loss of generality $r_i \notin D_i$), and D_{i+1} is obtained from D_i by moving one guard to r_i from an adjacent vertex $v \in D_i$. If the defender can win the game with the sets $\{D_i\}$, then each D_i is an *eternal dominating set*. The size of a smallest eternal dominating set of G is the *eternal domination number* $\gamma^\infty(G)$. This model, also referred to as the *one-guard moves* model, protects a graph against an infinite sequence of attacks executed one at a time and was first studied by Burger et al. in [5].

The **m-eternal domination problem**, which was introduced by Goddard et al. [10], is similar to the eternal domination problem, except that any number of guards may move in response to an attack (again occurring at a single vertex), as long as one guard moves to the attacked vertex. This model is also referred to as the *all-guards move* model of eternal domination. Other work on eternal and m-eternal domination include [1,2,8,9,11,12,16–22]. A recent survey of graph protection is given in [23].

Here we consider the **independent protection problem**, where each $D_i, i \geq 1$, is an independent dominating set, $r_i \in V - D_i$, and D_{i+1} is obtained from D_i by moving any number of guards, each across an edge, one of them to r_i . Thus one guard moves to the attacked vertex, and other guards move to maintain an independent dominating set. If an already occupied vertex is attacked, its guard deals with the attack without moving. The *independent protection number* $i^\infty(G)$ is the smallest number of guards required to protect the graph G against an arbitrary sequence of attacks, and if $i^\infty(G)$ exists, then G is said to be *independent-protectable*, abbreviated to *i-protectable*. For example, the cycle C_n is i-protectable: place guards on a maximal independent (i.e., independent dominating) set of cardinality $\lceil \frac{n}{3} \rceil$ and rotate the guards in an obvious way to deal with an attack at an unguarded vertex. Hence $i^\infty(C_n) = \lceil \frac{n}{3} \rceil$.

We also consider the **k-independent protection problem**, which is similar to the independent protection problem, except that several vertices of an independent set can be attacked simultaneously and any number of guards may move, one guard to each attacked vertex that is not already occupied, and other guards to maintain an independent set. For example, two guards on nonadjacent vertices of C_5 can handle simultaneous attacks on any two nonadjacent vertices repeatedly by rotating clockwise or counterclockwise. The *k-independent protection number* $i_k^\infty(G)$ is the smallest number of guards required to protect G against an arbitrary sequence of simultaneous attacks on up to k independent vertices, and if $i_k^\infty(G)$ exists, then G is said to be *k-i-protectable*. As another example, consider $C_7 = v_0, v_1, \dots, v_6, v_0$, let $D = \{v_0, v_2, v_4\}$, and place a guard on each vertex in D . Attack the vertices in an independent set S with $|S| = 3$. If $|S - D| = 1$, then the guards can be rotated to an independent set D' as in the case of a single attack. If $|S - D| = 2$, then by symmetry we may assume that $S - D = \{v_1, v_5\}$ or $\{v_1, v_6\}$, and either a clockwise or a counterclockwise rotation of the guards moves them to an independent set D' containing S . Similarly, if $D \cap S = \emptyset$, then (without loss of generality) $S = \{v_1, v_3, v_5\}$, and a suitable rotation of the guards to $D' = S$ deals with the attack. In each case there is an automorphism of C_7 that maps D to D' , hence this protection strategy can be repeated indefinitely. Hence C_7 is 3-i-protectable and $i_3^\infty(C_7) = 3$. We return to C_7 in Sections 3.1 and 3.2.

It is clear that if G is k -i-protectable for some integer k , then G is k' -i-protectable for each k' such that $1 \leq k' \leq k$, and $i_{k'}^\infty(G) \leq i_k^\infty(G)$. If G is k -i-protectable and S is a set of vertices such that guards placed on these vertices can protect G against any sequence of attacks, then S is called a *k-i-protecting set*.

Our aim is to consider the following problem.

Problem 1. Determine the classes of k -i-protectable graphs, $k \geq 1$.

After more definitions in Section 2 we consider existence and non-existence results in Section 3. We describe a colouring scheme as aid to determine whether a graph is k -i-protectable and to describe a protection strategy if it is. We show that well-covered graphs are α -i-protectable (Section 3.1), and non-well-covered graphs G are not $i(G)$ -i-protectable (Section 3.2). Bipartite graphs with perfect matchings are i-protectable, as are some without perfect matchings, and the differences between $i^\infty(G)$ and $i(G)$, and $\alpha(G)$ and $i^\infty(G)$, can be arbitrary (Section 3.3). Trees are i-protectable if and only if they have perfect matchings, and k -i-protectable for $k > 1$ if and only if they are well-covered (Section 3.4). The Petersen graph and other non-well-covered k -i-protectable graphs are discussed in Section 3.5. Methods for constructing larger k -i-protectable graphs from smaller ones are given in Section 4, and we close with open questions in Section 5.

2. Definitions and notation

We consider simple, undirected graphs $G = (V, E)$ and, in general, follow the notation of [6]. We denote the cardinalities of a smallest and a largest maximal independent set of vertices of G by $i(G)$ (the *independent domination number*) and $\alpha(G)$ (the *independence number*), respectively. It is well known that an independent set is maximal independent if and only if it is a dominating set [3]. A maximal independent set of cardinality $i(G)$ ($\alpha(G)$, respectively) is called an *i-set* (an α -set, respectively).

A graph is *well-covered* if $i(G) = \alpha(G)$, i.e., if all maximal independent sets have the same cardinality. The problem of characterizing well-covered graphs was proposed by Plummer [24] and was partly motivated by the observation that while

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