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Circumferences of 3-connected claw-free graphs

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ABSTRACT

In Li et al. (2009), proved that a 3-connected claw-free graph of order n with minimum degree δ contains a cycle of length at least $\min\{n, 6\delta - 15\}$, and they conjectured that such graphs should have a cycle of length at least $\min\{n, 9\delta - 3\}$. We prove that this conjecture is true with $\delta \geq 8$.

1. Introduction

In general, we shall use the notation of Bondy and Murty [1]. Graphs considered in this paper are finite and loopless. A graph is called a *multigraph* if it contains multiple edges. A graph without multiple edges is called a *simple graph* or simply a graph. As in [1], $\kappa'(G)$ and $d_G(v)$ denote the edge-connectivity of G and the degree of a vertex v in G, respectively. The minimum degree of a graph G is denoted by $\delta(G)$ or δ. An edge cut X of a graph G is *essential* if each of the components of $G - X$ has some edges. A graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut of size less than k. A vertex set $U \subseteq V(G)$ is called a *vertex cover* of G if every edge of G is incident with a vertex in U. The circumference of a graph G, denoted by $c(G)$, is the length of a longest cycle in G. A graph G of order n is Hamiltonian if $c(G) = n$. A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to $K_{1,3}$. In this paper, we will be concerned with the circumference of 3-connected claw-free graphs.

The following theorem of Dirac [6] is well known.

Theorem 1.1 (*Dirac [6]*). *Every 2-connected graph H of order n has $c(H) \geq \min\{n, 2\delta\}$.*

For 3-connected graphs and connected bipartite graphs, Voss and Zuluaga proved the following:

Theorem 1.2 (*Voss and Zuluaga [15]*).

(a) *If H is a 3-connected graph, then either $c(H) \geq 3\delta - 3$, or each longest cycle in H is a dominating cycle.*
(b) *If G is a 2-connected bipartite graph with bipartition $V(H) = A \cup B$, then $c(H) \geq \min\{4\delta - 4, 2|A|, 2|B|\}$.*

For claw-free graphs, Matthews and Sumner proved the following:

Theorem 1.3 (*Matthews and Sumner [10]*). *Every 2-connected claw-free graph H of order n has $c(H) \geq \min\{n, 2\delta + 4\}$.*

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Li, et al. proved the following theorem and posted a conjecture in [9]:

Theorem 1.4 (*Li, et al. [9]*). *Every* 3*-connected claw-free graph H of order n has* $c(H) \geq \min\{n, 6\delta - 15\}$.

Conjecture 1.5 (*Li, et al. Conjecture 1.8 [9]*). *Every* 3*-connected claw-free graph H of order n has* $c(H) \geq \min\{n, 9\delta - 3\}$.

The lower bound in Conjecture 1.5 stated in [9] was $9\delta - 6$. However, using the same example mentioned in [9] and described in [8], one can find that the correct best possible bound is $9\delta - 3$. In this paper, we prove Conjecture 1.5 for 3-connected claw-free graphs with $\delta \geq 8$.

Theorem 1.6. *Every* 3*-connected claw-free graph H of order n with $\delta \geq 8$ has* $c(H) \geq \min\{n, 9\delta - 3\}$.

The remainder of this paper is organized as follows. In Section 2, we give a brief discussion of Ryjáček's closure concept and Catlin's reduction method including the concepts of Catlin's reduced graphs and the core of essentially 3-edge-connected graphs. We show that Theorem 1.6 can be proved by reducing it to a problem of finding a graph with size at least $9\delta - 3$ that contains a dominating closed trail. In Section 3, we introduce some needed notation and prove some technical lemmas. In Section 4, we give an outline of the proof of the main theorem by dividing it into two associated theorems. The proofs of the associated theorems are given in the last two sections.

2. Ryjáček's closure concept and Catlin's reduction method

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. A trail T with end vertices x and y is called a *(x, y)-trail*. If $x = y$, then T is a closed trail, which is also called an *Eulerian graph*. A closed trail T in a graph G is called a *spanning closed trail* (SCT) of G if $V(G) = V(T)$ and is called a *dominating closed trail* (DCT) if $E(G - V(T)) = \emptyset$. A graph is *supereulerian* if it contains a spanning closed trail. The family of supereulerian graphs is denoted by $\mathcal{SL}$. The theorem below shows a relationship between closed trails in a graph and Hamiltonian cycles in its line graph.

Theorem 2.1 (*Harary and Nash-Williams [7]*). *The line graph $H = L(G)$ of a graph G with at least three edges is Hamiltonian if and only if G has a DCT.*

2.1. Ryjáček's closure concept

Ryjáček [12] defined the closure $cl(H)$ of a claw-free graph H to be one obtained by recursively adding edges to join two nonadjacent vertices in the neighborhood of any locally connected vertex of H as long as this is possible. A graph H is said to be *closed* if $H = cl(H)$.

Theorem 2.2. *(Ryjáček [12]) Let H be a claw-free graph and $cl(H)$ its closure. Then*
(a) *$cl(H)$ is well defined, and $\kappa(cl(H)) \geq \kappa(H)$;*
(b) *there is a triangle-free graph G such that $cl(H) = L(G)$;*
(c) *for every cycle C_0 in $L(G)$, there exists a cycle C in H with $V(C_0) \subseteq V(C)$.*

For a graph G, define

$$\overline{\sigma}_2(G) = \min\{d_G(u) + d_G(v) \mid \text{for every edge } uv \in E(G)\}. \quad (1)$$

If $cl(H) = L(G)$ is k-connected and $L(G)$ is not complete, then G is essentially k-edge-connected and $\delta(cl(H)) = \min\{d_G(x) + d_G(y) - 2 \mid xy \in E(G)\}$. Thus, $\overline{\sigma}_2(G) = \delta(cl(H)) + 2 \geq \delta(H) + 2$.

It is known that a connected line graph $H \neq K_3$ has a unique graph G with $H = L(G)$. We call G the preimage graph of H. For a claw-free graph H, the closure $cl(H)$ of H can be obtained in polynomial time [12] and the preimage graph of a line graph can be obtained in linear time [11]. Thus, we can compute G efficiently for $cl(H) = L(G)$. By Theorem 2.2, finding a cycle of length r in a claw-free graph H is reducible to finding a cycle of length r in its closure $cl(H) = L(G)$. By Theorem 2.1, this is equivalent to finding a subgraph Θ in G of size r that has a DCT. Therefore, to prove Theorem 1.6, it suffices to show the following holds.

Theorem 2.3. *Let G be an essentially* 3*-edge-connected graph with $n = |E(G)|$ and $\overline{\sigma}_2(G) \geq 10$. Then G has a subgraph Θ that has a DCT and has $|E(\Theta)| \geq \min\{n, 9\overline{\sigma}_2(G) - 21\}$.*

2.2. Catlin's reduction method

Let G be a connected multigraph. For $X \subseteq E(G)$, the *contraction* G/X is the multigraph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. Note that even G is a simple graph, multiple edges may arise by the identification. If Γ is a connected subgraph of G, we write G/Γ for $G/E(\Gamma)$ and say that G/Γ is obtained from G by contracting Γ.