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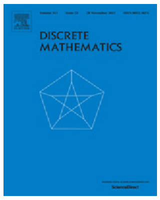

Bounded quantifier depth spectra for random graphs

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For which α there are first order graph statements A of given quantifier depth k such that a Zero–One law does not hold for the random graph $G(n, p(n))$ with $p(n)$ at or near (there are two notions) $n^{-\alpha}$? A fairly complete description is given in both the near dense (α near zero) and near linear (α near one) cases.

1. Introduction

Asymptotic behavior of first-order properties probabilities of the Erdős–Rényi random graph $G(n, p)$ have been widely studied in [7,3,12,9,8,13,16,15,17,18,27,11,21,20,22,24–26] (especially, the surveys [18,27] contain a description of all the main respective results). In [13] Shelah and the senior author showed that when α is an *irrational* number and $p(n) = n^{-\alpha+o(1)}$ then $G(n, p)$ obeys a Zero–One Law. (To avoid trivialities we shall restrict ourselves to $0 < \alpha < 1$.) In a series of papers [21,20,22,24–26] the junior author has examined when there is a Zero–One Law for all first order sentences of quantifier depth at most k. (In such cases we say that $G(n, p)$ *obeys Zero–One k-Law*.) We here consider two notions of spectra, relative to k.

We assume familiarity with the Erdős–Rényi random graph $G(n, p)$ and of threshold functions (see [27,10,4,1]). We further assume familiarity with the first order language for graphs (see [18,27,2,5,19]). The quantifier depth of a sentence L is the number of nested quantifiers [27,19]. We let $\mathcal{L}_k$ denote the set of sentences L with quantifier depth at most k.

As illustrative examples, the existence of a K_4 has threshold function $n^{-2/3}$. The property that every pair x_1, x_2 of vertices have a common neighbor y has threshold function $n^{-1/2}\sqrt{\ln n}$.

For any first order property L we define two notions of its spectra, $S^1(L)$ and $S^2(L)$. The first considers behavior *at* $p = n^{-\alpha}$. $S^1(L)$ is the set of $\alpha \in (0, 1)$ which do *not* satisfy the following property: With $p(n) = n^{-\alpha}$, $\lim_{n\to\infty} \Pr[G(n, p(n)) \models L]$ exists and is either zero or one. The second considers behavior *near* $p = n^{-\alpha}$. $S^2(L)$ is the set of $\alpha \in (0, 1)$ which do *not* satisfy the following property: There exists $\epsilon > 0$ so that for any $n^{-\alpha-\epsilon} < p(n) < n^{-\alpha+\epsilon}$, $\lim_{n\to\infty} \Pr[G(n, p(n)) \models L] = \delta$ exists, is either zero or one, and is independent of the choice of $p(n)$.

Tautologically $S^1(L) \subset S^2(L)$ but we need not have equality. Letting L be the sentence that every two vertices have a common neighbor, $S^2(L) = \{\frac{1}{2}\}$ while $S^1(L) = \emptyset$.

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Definition 1. Let $k \geq 1$. S_k^1 is the union of all $S^1(L)$ where $L \in \mathcal{L}_k$. S_k^2 is the union of all $S^2(L)$ where $L \in \mathcal{L}_k$.

A full description of S_k^1 and S_k^2 appears difficult. Our main (though not only) concern shall be the values α of S_k^1 and S_k^2 that lie either near zero or near one.

2. Previous results

Theorem 2 ([13]). *Every $S^2(L)$ consists only of rational values α (as $S^1(L) \subseteq S^2(L)$, the same is true for $S^1(L)$). Moreover, $\bigcup_{L\in\mathcal{L}} S^1(L) = \mathbb{Q} \cap (0, 1)$.*

In [21,20,22,24,25] some rational points from the set $(0, 1) \setminus S_k^1$ were obtained.

Theorem 3 ([21]). *Let $k \geq 3$ be an arbitrary natural number. If $\alpha \in (0, \frac{1}{k-2})$ then the random graph $G(n, n^{-\alpha})$ obeys Zero–One k-Law. Moreover, $\frac{1}{k-2} \in S_k^1$.*

From this result it follows that the minimal number in S_k^1 equals $\frac{1}{k-2}$. We also obtain the maximal number in S_k^1.

Theorem 4 ([22]). *Let $k > 3$ be an arbitrary natural number. Let $\mathcal{Q}$ be the set of positive rational numbers with the numerator less than or equal to 2^{k-1}. The random graph $G(n, n^{-\alpha})$ obeys the Zero–One k-Law, if $\alpha = 1 - \frac{1}{2^{k-1}+\beta}$, $\beta \in (0, \infty) \setminus \mathcal{Q}$. Moreover, for any $\beta \in \{1, \ldots, 2^{k-1} - 2\}$*

$$1 - \frac{1}{2^{k-1} + \beta} \in S_k^1.$$

Note that this result implies the following statement. For any $k > 3$, $\alpha > 1 - \frac{1}{2^{k-2}}$, the random graph $G(n, n^{-\alpha})$ obeys the Zero–One k-Law, if $\alpha \notin \{1 - \frac{1}{2^k}, 1 - \frac{1}{2^k-1}\}$. However, the maximal α such that $G(n, n^{-\alpha})$ obeys the Zero–One k-Law is known.

Theorem 5 ([24]). *Let $k > 3$ be an arbitrary natural number. Moreover, let $\alpha \in \{1 - \frac{1}{2^k}, 1 - \frac{1}{2^k-1}\}$. Then the random graph $G(n, n^{-\alpha})$ obeys the Zero–One k-Law.*

Hence the maximal number in S_k^1 equals $1 - \frac{1}{2^k-2}$.

Recently, we extend the subset of the set $\mathcal{Q}$ from Theorem 4 such that for any β from this subset $1 - \frac{1}{2^{k-1}+\beta} \in S_k^1$.

Theorem 6 ([26]). *Let $k > 4$ be an arbitrary natural number. Moreover, let $\alpha = 1 - \frac{1}{2^{k-1}+\beta}$, where $\beta = \frac{a}{b}$ is an irreducible positive fraction with $a \in \{1, 2, \ldots, 2^{k-1} - (b+1)^2\}$. Then $\alpha \in S_k^1$.*

In [15] it was proved that sets S_k^1 and S_k^2 are infinite when k is large enough.

Theorem 7 ([15]). *There exists k_0 such that for any natural $k > k_0$ sets S_k^1 and S_k^2 are infinite.*

There are, up to tautological equivalence, (see, e.g., [19]) only a finite number of first order sentences of a given quantifier depth. Thus, for j either 1 or 2, set S_k^j is infinite if and only if there is a single L of quantifier depth at most k such that $S^j(L)$ is infinite. Therefore, we always search for one property with infinite spectrum when we prove that the spectrum S_k^j is infinite.

It is also known [17] that all limit points of S_k^1 and S_k^2 are approached only from above.

Theorem 8 ([17]). *For any $k \in \mathbb{N}$ the set S_k^2 is well-ordered under $>$.*

Consequently, the set S_k^1 follows the same property.

In this paper we try to answer the following questions.

Q1 What are the maximal and the minimal numbers in S_k^2?

Q2 Let k be large enough so that sets S_k^1 and S_k^2 are infinite. What are the maximal and the minimal limit points in S_k^1 and S_k^2?

Q3 How many elements are there in S_k^1 and S_k^2 near their minimal elements (the answer on this question for the maximal elements is given in Theorem 6: $\left|S_k^j \cap \left(1 - \frac{1}{2^{k-1}}, 1\right)\right| = \Omega(2^{3k/2})$ for $j \in \{1, 2\}$)? Consider, say, the interval $I = \left(0, \frac{1}{k-2.5}\right)$. How many elements are there in $S_k^j \cap I$, $j \in \{1, 2\}$?

Q4 For each $j \in \{1, 2\}$ what is the minimal k such that S_k^j is infinite?

3. New results

For any natural k we find the maximal and the minimal numbers in S_k^2 and, therefore, answer the question Q1.

Theorem 9. *If $k > 3$, then $\min S_k^2 = \frac{1}{k-1}$, $\max S_k^2 = 1 - \frac{1}{2^{k-2}}$. Moreover, $S_3^2 = \{\frac{1}{2}, \frac{2}{3}\}$.*