Discrete Mathematics 339 (2016) 1665–1676

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Total dominating sequences in graphs

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ARTICLE INFO

Article history:
Received 27 July 2015
Received in revised form 12 January 2016
Accepted 13 January 2016
Available online 17 February 2016

Keywords:
Total domination
Edge cover
Grundy total domination number

ABSTRACT

A vertex in a graph totally dominates another vertex if they are adjacent. A sequence of vertices in a graph G is called a total dominating sequence if every vertex v in the sequence totally dominates at least one vertex that was not totally dominated by any vertex that precedes v in the sequence, and at the end all vertices of G are totally dominated. While the length of a shortest such sequence is the total domination number of G, in this paper we investigate total dominating sequences of maximum length, which we call the Grundy total domination number, $\gamma_{gr}^t(G)$, of G. We provide a characterization of the graphs G for which $\gamma_{gr}^t(G) = |V(G)|$ and of those for which $\gamma_{gr}^t(G) = 2$. We show that if T is a nontrivial tree of order n with no vertex with two or more leaf-neighbors, then $\gamma_{gr}^t(T) \geq \frac{2}{3}(n+1)$, and characterize the extremal trees. We also prove that for $k \geq 3$, if G is a connected k-regular graph of order n different from $K_{k,k}$, then $\gamma_{gr}^t(G) \geq (n + \lceil \frac{k}{2} \rceil - 2)/(k-1)$ if G is not bipartite and $\gamma_{gr}^t(G) \geq (n + 2\lceil \frac{k}{2} \rceil - 4)/(k-1)$ if G is bipartite. The Grundy total domination number is proven to be bounded from above by two times the Grundy domination number, while the former invariant can be arbitrarily smaller than the latter. Finally, a natural connection with edge covering sequences in hypergraphs is established, which in particular yields the NP-completeness of the decision version of the Grundy total domination number.

1. Introduction

The concept of edge covering sequences was introduced in [3] to shed more light on the possible procedures of determining the edge cover number of a hypergraph (*edge cover number* is the cardinality of a smallest set of (hyper)edges in a hypergraph whose union equals the set of its vertices). Of particular interest is the maximum length of a sequence, in which one only uses the most basic greedy condition that each edge must contain a vertex that is not contained in the edges that precede it, and is called the Grundy covering number of a hypergraph. (The name arises from the Grundy coloring number, which is the maximum number of colors that are used in a greedy coloring algorithm. The concept of Grundy colorings was introduced back in the 1970s [5] and has been investigated in many papers.) In [3] the main focus was on dominating sequences (of vertices) in graphs, which can be viewed precisely as edge covering sequences of the hypergraph of closed neighborhoods of the graph. The longest possible dominating sequences were determined in several classes of graphs (e.g. trees, split graphs, cographs), while it was shown that this problem is NP-complete, even when restricted to chordal graphs [3].

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http://dx.doi.org/10.1016/j.disc.2016.01.017

In this paper we introduce and investigate total dominating sequences in graphs, which arise from the hypergraph of open neighborhoods of a graph. Total domination is one of the classical concepts in graph theory, having numerous applications and connections with other parameters. It was recently surveyed in the monograph [12]. The *total domination number*, $\gamma_t(G)$, of a graph G with no isolated vertices is the smallest cardinality of a set of vertices S such that every vertex of G has a neighbor in S. (If the condition only requires that vertices from $V(G) \setminus S$ have a neighbor in S, then the resulting invariant is the *domination number* $\gamma(G)$ of G.) It is well-known that for every graph G with no isolated vertices we have $\gamma(G) \le \gamma_t(G) \le 2\gamma(G)$. One of the central problems in this area is to determine good upper bounds for the total domination number of a graph in terms of its order. Cockayne, Dawes, and Hedetniemi [7] showed that if G is connected of order $n \ge 3$, then $\gamma_t(G) \le \frac{2}{3}n$. Several authors [1,6,18] showed that if G is a graph of order n with minimum degree at least 3, then $\gamma_t(G) \le \frac{1}{2}n$. Thomassé and Yeo [17] showed that if G is a graph of order n with minimum degree at least 4, then $\gamma_t(G) \le \frac{3}{7}n$.

We now introduce our main invariant, which is defined for all graphs G without isolated vertices. Let $S = (v_1, \dots, v_k)$ be a sequence of distinct vertices of G. The corresponding set $\{v_1, \dots, v_k\}$ of vertices from the sequence S will be denoted by $\widehat{S}$. The sequence S is a *legal (open neighborhood) sequence* if

$$N(v_i) \setminus \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset \tag{1}$$

holds for every $i \in \{2, \dots, k\}$. If, in addition, $\widehat{S}$ is a total dominating set of G, then we call S a *total dominating sequence* of G. If S is a legal sequence, then we will say that v_i *footprints* the vertices from $N(v_i) \setminus \cup_{j=1}^{i-1} N(v_j)$, and that v_i is the *footprinter* of every vertex $u \in N(v_i) \setminus \cup_{j=1}^{i-1} N(v_j)$. That is, v_i footprints vertex u if v_i totally dominates u, and u is not totally dominated by any of the vertices that precede v_i in the sequence. Thus the function $f_S : V(G) \to \widehat{S}$ that maps each vertex to its footprinter is well defined. Clearly the length k of a total dominating sequence S is bounded from below by the total domination number, $\gamma_t(G)$, of G. On the other hand, the maximum length of a total dominating sequence in G will be called the *Grundy total domination number* of G and will be denoted by $\gamma_{gr}^t(G)$. The corresponding sequence will be called a *Grundy total dominating sequence* of G.

The paper is organized as follows. In the next section we fix the notation and state some preliminary results and observations. In particular we prove an upper bound for the Grundy total domination number in terms of the order and minimum degree of a graph, and a lower bound in terms of the order and maximum degree. Section 3 considers two total domination chains that arise from some invariants related to the Grundy total domination number, notably the total domination number, the game total domination number, and the upper total domination number. In Section 4 we characterize two extremal families of graphs, that is, the graphs whose Grundy total domination number is equal to 2, and the graphs whose Grundy total domination number is equal to their order. While the former are exactly complete multipartite graphs, the latter family can only be described in a more involved fashion, which in the class of trees reduces to exactly the trees having a perfect matching; this result is established in Section 5. This section also contains the proof of the lower bound $\gamma_{gr}^t(T) \ge \frac{2}{3}(n+1)$, where T is an arbitrary tree, together with the characterization of the trees attaining this bound. Section 6 contains our most involved result, which is the lower bound for the Grundy total domination number of regular graphs, when complete bipartite graphs are excluded. In Section 7 the bounds between the Grundy total domination number and the Grundy domination number are discussed, while Section 8 connects the new concept with edge covering sequences of hypergraphs. As a result of these connections, we first establish the existence of total dominating sequences in G of arbitrary length between $\gamma_t(G)$ and $\gamma_{gr}^t(G)$, and then we prove the NP-completeness of the corresponding Grundy total domination problem. We conclude in the last section with some open problems that arise throughout the paper.

2. Notation and preliminary results

For notation and graph theory terminology, we in general follow [12]. We assume throughout the remainder of the paper that all graphs considered are *without isolated vertices*. The *degree* of a vertex v in G, denoted $d_G(v)$, is the number of neighbors, $|N_G(v)|$, of v in G. The minimum and maximum degree among all the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *leaf* is a vertex of degree 1, while its neighbor is a *support vertex*. A *strong support vertex* is a vertex with at least two leaf-neighbors. The subgraph induced by a set S of vertices of G is denoted by $G[S]$. A *non-trivial graph* is a graph on at least two vertices.

A *cycle* on n vertices is denoted by C_n and a *path* on n vertices by P_n. A *star* is a tree $K_{1,n}$ for some $n \ge 1$. A *complete k-partite graph* is a graph that can be partitioned into k independent sets, so that every pair of vertices from two different independent sets is adjacent. A *complete multipartite graph* is a graph that is complete k-partite for some k. In particular, complete bipartite and complete graphs are in the family of complete multipartite graphs.

Two distinct vertices u and v of a graph G are *open twins* if $N(u) = N(v)$. A graph is *open twin-free* if it has no open twins. We remark that a tree is open twin-free if and only if it has no strong support vertex.

A *total dominating set* of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S; that is, every vertex has a neighbor in S. If we only require that every vertex outside S has a neighbor in S, then S is called a *dominating set* of G. The *upper total domination number*, $\Gamma_t(G)$, of G is the maximum cardinality of a minimal total dominating set in G.