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# An extremal problem for vertex partition of complete multipartite graphs

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ABSTRACT

For a graph $G$ and a family $\mathcal{H}$ of graphs, a vertex partition of $G$ is called an $\mathcal{H}$-decomposition, if every part induces a graph isomorphic to one of $\mathcal{H}$. For $1 \leq a \leq k$, let $A(k, a)$ denote the graph which is a join of an empty graph of order $a$ and a complete graph of order $k - a$. Let $\mathcal{A}_k = \{A(k, a) : 1 \leq a \leq k\}$. In this paper, extremal problems related to $\mathcal{H}$-decomposition of a complete multipartite graph, where $\mathcal{H} \subset \mathcal{A}_k$, are studied. Among other results, it is proved that for every complete multipartite graph $G$ of order $k\ell$, where $\ell \geq k - 2 \geq 2$, there is a positive integer $a$ such that $G$ admits an $\{A(k, a), A(k, a + 1), A(k, a + 2)\}$-decomposition.



## 1. Introduction

A graph is finite and undirected with no multiple edges or loops. Let $\mathcal{H}$ be a family of graphs. For a graph $G$, we call a vertex partition $V(G) = V_1 \cup \cdots \cup V_\ell$ an *$\mathcal{H}$-decomposition*, if $G[V_i] \in \mathcal{H}$ for all $1 \leq i \leq \ell$, where $G[V_i]$ is a subgraph of $G$ induced by $V_i$. In the following, we mainly consider the case where $G$ is a complete multipartite graph and $\mathcal{H}$ consists of graphs with a common number of vertices.

Our aim is to find sufficient conditions for the existence of an $\mathcal{H}$-decomposition having some nice properties. The problems raised in the paper can be considered as a coin problem. Let a set of piles of coins be given. A pile of coins corresponds to a partite set of complete multipartite graph, and a rearrangement of coins corresponds to a vertex partition of a complete multipartite graph.

The next results were proved in [3,4].

**Theorem A** ([3]). *For every complete multipartite graph $G$ of order $(k+1)\ell - 1$, where $\ell \geq k - 2$, there is an induced subgraph $G'$ of order $k\ell$ such that $G'$ admits an $\{H\}$-decomposition with some complete multipartite graph $H$ of order $k$.*

**Theorem B** ([4]). *For every complete multipartite graph $G$ of order $k\ell$, where $k \geq 2$ and $\ell \geq 2$, there is a pair of complete multipartite graphs $H_1, H_2$ of order $k$ such that $G$ admits an $\{H_1, H_2\}$-decomposition.*

In the following, $K_{n_1,n_2,\ldots,n_s}$ is denoted by $(n_1, n_2, \ldots, n_s)$. Furthermore, if $t$ partite sets have a common order $a$, we write as $(\ldots, a^t, \ldots)$ instead of $(\ldots, a, a, \ldots, a, \ldots)$. In particular, $(n)$ denotes the empty graph of order $n$, and $(1^n)$ denotes the complete graph of order $n$.

Let $\mathcal{A}_k = \{(a, 1^{k-a}) : 1 \leq a \leq k\}$. $\mathcal{A}_k$ is a family of graphs of order $k$ which consists of a complete graph $(1^k)$, and an empty graph $(k)$ and joins of a complete graph and an empty graph. In this paper, we focus on an $\mathcal{H}$-decomposition, where $\mathcal{H} \subset \mathcal{A}_k$.

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## 2. Main results

In this section, we present the main results of the paper. The proofs will be given in Section 3. Firstly, we have the following result.

**Theorem 1.** *For every complete multipartite graph $G$ of order $k\ell$, where $\ell \geq k-2 \geq 2$, there is a positive integer $a$ such that $G$ admits an $\{(a, 1^{k-a}), (a+1, 1^{k-(a+1)}), (a+2, 1^{k-(a+2)})\}$-decomposition.*

The next statement is an immediate consequence of Theorem 1.

**Corollary 2.** *Every complete multipartite graph $G$ of order $k\ell$, where $\ell \geq k-2 \geq 2$, admits an $\mathcal{A}_k$-decomposition.*

The bound for $\ell$ in Corollary 2 (and also in Theorem 1) is tight. To see this, consider the complete bipartite graph $G = ((k-1)\ell - 1, k-2)$ with $\ell = k-3$ and $k \geq 4$. Then $G$ has order $k\ell$, but $G$ has no $\mathcal{A}_k$-decomposition. Assume to the contrary that $G$ has an $\mathcal{A}_k$-decomposition. Then $G$ has a $\{(k), (k-1, 1)\}$-decomposition, since $G$ has only two partite sets and contains no copy of $(a, 1^{k-a})$ for $1 \leq a \leq k-2$. However, this is impossible, since $G$ has at most $\ell - 1$ vertex disjoint copies of $(k-1)$.

A related result of Theorem 1 is as follows.

**Theorem 3.** *Let $G$ be a complete multipartite graph of order $k\ell$. Then the following statements hold:*
(a) *If $k = 3$, then $G$ has a $\{(2, 1), (3)\}$-decomposition or $\{(1^3), (2, 1)\}$-decomposition.*
(b) *If $k \geq 4$ and $\ell \geq 2k-6$, then there is a positive integer $a$ such that $G$ admits an $\{(a, 1^{k-a}), (a+1, 1^{k-(a+1)})\}$-decomposition.*

The bound for $\ell$ in Theorem 3(b) is tight. To see this, consider the complete multipartite graph $G = ((k-1)(k-3)-1, (k-1)(k-3)-1, k-4)$ with $\ell = 2k-7$ and $k \geq 4$. Then $G$ has order $k\ell$, but $G$ has no $\{(a, 1^{k-a}), (a+1, 1^{k-(a+1)})\}$-decomposition. For $k = 4$, then $G = (2, 2)$ has clearly no $\mathcal{A}_k$-decomposition. For $k \geq 5$, let $P_1$, $P_2$ and $P_3$ be three partite sets of $G$ with $|P_1| = |P_2| = (k-3)(k-1) - 1$ and $|P_3| = k-4$. Suppose to the contrary that $G$ has an $\{(a, 1^{k-a}), (a+1, 1^{k-(a+1)})\}$-decomposition, Since $G$ has only three partite sets, we have $a \geq k-2$. Furthermore, since $G$ contains at most $\ell - 1$ vertex disjoint copies of $(k-1)$, we have $a = k-2$. Without loss of generality, we may assume $P_1$ is partitioned into $k-3$ copies of $(k-2)$ and $P_2$ is partitioned into $k-4$ copies of $(k-2)$. However, this is impossible, since the number of the remaining vertices of $P_2$ is $(k-1)(k-3) - 1 - (k-4)(k-2) = 2k-6$, which is greater than $\ell$.

If every complete multipartite graph of order $k\ell$ admits an $\mathcal{H}$-decomposition, where $\mathcal{H}$ is a family of graphs each of which has order $k$, then we need $\{(k), (k-1, 1)\} \subset \mathcal{H}$ for $G = (k\ell - 1, 1)$ and also need $(1^k) \in \mathcal{H}$ for $G = (1^{k\ell})$. Conversely, these three graphs $(k)$, $(k-1, 1)$, $(1^k)$ suffice when $\ell$ is sufficiently large.

**Theorem 4.** *Every complete multipartite graph of order $k\ell$, where $k \geq 4$ and $\ell \geq (k-2)^2$, admits a $\{(k), (k-1, 1), (1^k)\}$-decomposition.*

The bound for $\ell$ in Theorem 4 is tight. To see this, consider the multipartite graph $G = ((k-1)\ell - 1, (k-2)^{k-2})$ with $\ell = (k-2)^2 - 1$ and $k \geq 4$. Then $G$ has order $k\ell$, but $G$ has no $\{(k), (k-1, 1), (1^k)\}$-decomposition. Assume to the contrary that $G$ has a $\{(k), (k-1, 1), (1^k)\}$-decomposition. Then $G$ has a $\{(k), (k-1, 1)\}$-decomposition, since $G$ has no copy of $(1^k)$. However, this is impossible, since $G$ has at most $\ell - 1$ vertex disjoint copies of $(k-1)$.

**Theorem 5.** *Let $G$ be a complete multipartite graph of order $k\ell$. Then the following statements hold:*
(a) *If $k = 3$, then $G$ has a $\{(3), (2, 1)\}$-decomposition or a $\{(3), (1^3)\}$-decomposition.*
(b) *If $k \geq 4$ and $\ell \geq \frac{1}{2}(3k^2 - 9k + 4)$, then $G$ has a $\{(k), (k-1, 1)\}$-decomposition or a $\{(k), (1^k)\}$-decomposition.*

The bound for $\ell$ in Theorem 5(b) is tight. To see this, consider the multipartite graph $G = (k(k-1)(k-3)-1, (k-1)^2-1, (k-2)(k-1)-1, (k-3)(k-1)-1, \ldots, 2(k-1)-1, 1^{(k^2-k-4)/2})$ with $\ell = (1/2)(3k^2 - 9k + 2)$ and $k \geq 4$. Then $G$ has order $k\ell$, but $G$ has neither $\{(k), (k-1, 1)\}$-decomposition nor $\{(k), (1^k)\}$-decomposition.

Firstly, we will show that $G$ has no $\{(k), (k-1, 1)\}$-decomposition. The maximum number of vertex disjoint copies of $(k-1)$ in $G$ is

$$k(k-3) - 1 + \sum_{i=1}^{k-2} i = \frac{1}{2}(3k^2 - 9k) < \ell.$$

Hence, $G$ has no $\{(k), (k-1, 1)\}$-decomposition.

Next, we will show that $G$ has no $\{(k), (1^k)\}$-decomposition. Note that the maximum number of vertex disjoint copies of $(k)$ in $G$ is

$$(k-1)(k-3) - 1 + \sum_{i=1}^{k-2} i = \frac{1}{2}(3k^2 - 11k + 6) = \ell - k + 2.$$