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The maximum number of subset divisors of a given size

Samuel Zbarsky

Carnegie Mellon University, United States

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ABSTRACT

If s is a positive integer and A is a set of positive integers, we say that B is an s-divisor of A if $\sum_{b\in B} b \mid s\sum_{a\in A} a$. We study the maximal number of k-subsets of an n-element set that can be s-divisors. We provide a counterexample to a conjecture of Huynh that for $s = 1$, the answer is $\binom{n-1}{k}$ with only finitely many exceptions, but prove that adding a necessary condition makes this true. Moreover, we show that under a similar condition, the answer is $\binom{n-1}{k}$ with only finitely many exceptions for each s.

1. Introduction

If X is a set of positive integers, let $\sum X$ denote $\sum_{x\in X} x$. Let A be a finite subset of the positive integers. The elements of A are $a_1 < a_2 < \cdots < a_n$ and let B be a subset of A. We say that B is a *divisor* of A if $\sum B \mid \sum A$. We define $d_k(A)$ to be the number of k-subset divisors of A and let $d(k, n)$ be the maximum value of $d_k(A)$ over all sets A of n positive integers.

Similarly, for $s \geq 1$ a positive integer, we say that B is an *s-divisor* of A if $\sum B \mid s\sum A$. We define $d_k^s(A)$ to be the number of k-subset s-divisors of A and let $d^s(k, n)$ be the maximum value of $d_k^s(A)$ over all sets A of n positive integers.

Note that the concepts of divisor and 1-divisor coincide. Also, if B is a divisor of A, then B is an s-divisor of A for all s, so $d_k^s(A) \geq d_k(A)$ and $d^s(k, n) \geq d(k, n)$.

Huynh [6] notes that for all values of $a_1, \ldots, a_{n-1}$, we can pick an a_n and set $A = \{a_1, \ldots, a_{n-1}, a_n\}$ so that every k-subset of $\{a_1, \ldots, a_{n-1}\}$ is an A-divisor. Therefore $d(k, n) \geq \binom{n-1}{k}$ for all $1 \leq k \leq n$. This motivates the definition that A is a *k-anti-pencil* if the set of k-subset divisors of A is $\binom{A\setminus\{a_n\}}{k}$. We similarly define A to be a *(k, s)-anti-pencil* if the set of k-subset s-divisors of A is $\binom{A\setminus\{a_n\}}{k}$.

Huynh [6] also formulates the following conjecture (Conjecture 22).

Conjecture 1. *For all but finitely many values of k and n,* $d(k, n) = \binom{n-1}{k}$.

In this paper, we provide infinite families of counterexamples, but prove that, with the exception of these families, the conjecture is true. This gives us the following modified form.

Conjecture 2. *For all but finitely many integer pairs* (k, n) *with* $1 < k < n$, $d(k, n) = \binom{n-1}{k}$.

For convenience, we now rescale, dividing every element of A by $\sum A$, so that now the elements of A are positive rational numbers and $\sum A = 1$. Under this rescaling, $B \subseteq A$ is a divisor of A if and only if $\sum B = \frac{1}{m}$ for some positive integer m and B is an s-divisor of A if and only if $\sum B = \frac{s}{m}$ for some positive integer m. Clearly, the values of $d(k, n)$ and $d^s(k, n)$ do not change.

E-mail address: sa_zbarsky@yahoo.com.

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The $k < n$ condition in Conjecture 2 is necessary since it is easy to see that $d(n, n) = 1 > \binom{n-1}{n}$. Also, if

$$A = \left\{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^{n-2}}, \frac{1}{3(2^{n-2})}, \frac{1}{3(2^{n-3})}\right\}$$

then $\sum A = 1$, so $d_1(A) = n$ and $d(1, n) = n > \binom{n-1}{1}$. Therefore the $1 < k$ condition is necessary.

However, we prove that these families of values (k, n) cover all but finitely many exceptions.

Theorem 3. *For all but finitely many pairs (k, n), if $1 < k < n$, $|A| = n$, and $d_k(A) \geq \binom{n-1}{k}$, then A is a k-anti-pencil.*

Note that this immediately implies Conjecture 2.

If we are interested in s-divisors, we get another family of exceptions. If $s \geq 2$, $a_{n-1} = \frac{1}{s+1}$ and $a_n = \frac{2}{s+2}$, then $d^s_{n-1}(A) \geq 2$, so $d^s(n-1, n) \geq 2 > \binom{n-1}{n-1}$. However, we prove that these cover all but finitely many exceptions.

Theorem 4. *Fix $s \geq 1$. For all but finitely many pairs (k, n) (with the number of these pairs depending on s), if $1 < k < n - 1$, $|A| = n$, and $d^s_k(A) \geq \binom{n-1}{k}$, then A is a (k, s)-anti-pencil.*

Note that this immediately implies the following corollary.

Corollary 5. *Fix $s \geq 1$. Then $d^s(k, n) = \binom{n-1}{k}$ for all but finitely many pairs (k, n) with $1 < k < n - 1$ (with the number of these pairs depending on s).*

We will prove Theorem 4. In the $s = 1$ case, where $k = n - 1$, if $i \leq n - 1$, then $\sum(A\backslash\{a_i\}) > \frac{1}{2}$, so $A\backslash\{a_i\}$ is not a divisor of A. This, together with the $s = 1$ case of Theorem 4, gives us Theorem 3.

2. Lemmas

We will need a lemma about a certain poset. First, we present some general definitions and theorems (all the definitions and results up to the lemma statement can be found in [4]).

The *width* of a poset is the size of its largest antichain. If P is a finite poset, we say that P is *ranked* if there exists a function $\rho : P \to \mathbb{Z}$ satisfying $\rho(y) = \rho(x) + 1$ if y covers x in P (i.e. $y > x$, and there is no $z \in P$ with $y > z > x$). If $\rho(x) = i$, then x is said to have *rank* i. Let P_i denote the set of elements of P of rank i. We say P is *rank-symmetric rank-unimodal* if there exists some $c \in \mathbb{Z}$ with $|P_i| \leq |P_{i+1}|$ when $i < c$ and $|P_{2c-i}| = |P_i|$ for all $i \in \mathbb{Z}$. A ranked poset P is called *strongly Sperner* if for every positive integer s, the largest subset of P that has no $(s + 1)$-chain is the union of the s largest P_i.

Proctor, Saks, and Sturtevant [9] prove that the class of rank-symmetric rank-unimodal strongly Sperner posets is closed under products. A finite product of finite linear orders is called a *chain product*. Since a linear ordering of length n is rank-symmetric rank-unimodal strongly Sperner, so is any chain product.

We now take a d-dimensional lattice cube with n lattice points per edge. Define a poset on the lattice points by $(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d)$ if $x_i \leq y_i$ for all i. This is a chain product, so it is rank-symmetric rank-unimodal strongly Sperner.

Lemma 6. *The largest antichain in this poset has at most $(n + d - 2)^{d-1}\sqrt{\frac{2}{d}}$ elements.*

Proof. Center the cube on the origin by translation in $\mathbb{R}^d$. Let U be the set of elements whose coordinates sum to 0. Since the poset is rank-symmetric rank-unimodal strongly Sperner, its width is at most the size of P_c, which is $|U|$.

For each $y = (y_1, \ldots, y_d) \in U$, let S_y be the set of points $(x_1, \ldots, x_d)$ with $|x_i - y_i| < \frac{1}{2}$ for $1 \leq i \leq d - 1$ (note that this does not include the last index) which lie on the hyperplane given by $x_1 + \cdots + x_d = 0$. If y, z are distinct elements of U, then S_y and S_z are clearly disjoint. Also, the projection of S_y onto the hyperplane given by $x_d = 0$ is a unit $(d - 1)$-dimensional hypercube, which has volume 1. Thus the volume of S_y is $\sqrt{d}$ and the volume of $\bigcup_{y\in U} S_y$ is $|U|\sqrt{d}$.

On the other hand, if $(x_1, \ldots, x_d) \in S_y$, then $|x_i - y_i| < \frac{1}{2}$ for $1 \leq i \leq d - 1$ and $|x_d - y_d| \leq \sum_{i=1}^{d-1} |x_i - y_i| < \frac{1}{2}(d - 1)$. Thus $(x_1, \ldots, x_d)$ lies in the cube of edge length $(n - 1) + (d - 1) = n + d - 2$ centered at the origin. Therefore $\bigcup_{y\in U} S_y$ lies in the intersection of a cube of edge length $n + d - 2$ with a hyperplane through its center (the origin).

Ball [1] shows that the volume of the intersection of a unit hypercube of arbitrary dimension with a hyperplane through its center is at most $\sqrt{2}$. Therefore the volume of $\bigcup_{y\in U} S_y$ is at most $(n + d - 2)^{d-1}\sqrt{2}$, so

$$|U| \leq (n + d - 2)^{d-1}\sqrt{\frac{2}{d}}. \quad \square$$

Let $X = \{x_1 < \cdots < x_n\}$ be a set of positive integers. If $B, C \in \binom{X}{d}$, then we say that $B \leq C$ if we can write $B = \{b_1, \ldots, b_d\}$ and $C = \{c_1, \ldots, c_d\}$ with $b_i \leq c_i$ for all $1 \leq i \leq d$. Whenever we compare subsets of A, we will be using this partial order.