Discrete Mathematics 339 (2016) 1744–1752

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The clique number and the smallest Q-eigenvalue of graphs

Leonardo de Lima [a], Vladimir Nikiforov [b,*], Carla Oliveira [c]

[a] Production Engineering Department, Federal Center of Technological Education Celso Suckow da Fonseca, Rio de Janeiro, Brazil
[b] Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[c] Mathematics Department and Graduate Program of Technology, National School of Statistics, Rio de Janeiro, Brazil

ARTICLE INFO

Article history:
Received 7 August 2015
Received in revised form 29 January 2016
Accepted 1 February 2016
Available online 26 February 2016

Keywords:
Blow-up graphs
Laplacian
Signless Laplacian
Complete subgraphs
Extremal problem
Clique number

ABSTRACT

Let $q_{\min}(G)$ stand for the smallest eigenvalue of the signless Laplacian of a graph G of order n. This paper gives some results on the following extremal problem:

How large can $q_{\min}(G)$ be if G is a graph of order n, with no complete subgraph of order $r+1$?

It is shown that this problem is related to the well-known topic of making graphs bipartite. Using known classical results, several bounds on $q_{\min}$ are obtained, thus extending previous work of Brandt for regular graphs.

In addition, the spectra of the Laplacian and the signless Laplacian of blowups of graphs are calculated. Finally, using graph blowups, a general asymptotic result about the maximum $q_{\min}$ is established.

1. Introduction

In this paper we study how large can the smallest signless Laplacian eigenvalue of graphs with bounded clique number be. Arguably the most attractive problems in spectral graph theory are the extremal ones, with general form like:

If G is a graph of order n, with some property $\mathcal{P}$, how large can its kth eigenvalue be?

From this general template, by choosing the property $\mathcal{P}$, the type of graph matrix, and the value k, we can obtain an amazing variety of concrete spectral problems, ranging from trivial to extremely challenging ones. The study of such extremal questions is crucial to graph theory, for they provide a sure way to connect the structure of a graph to its eigenvalues.

In this vein, we shall introduce a new extremal problem about the smallest signless Laplacian eigenvalue of graphs with no complete subgraphs of given order.

First, recall a few definitions: Given a graph G, write A for the adjacency matrix of G and let D be the diagonal matrix of the degrees of G. The *Laplacian* $L(G)$ and the *signless Laplacian* $Q(G)$ of G are defined as $L(G) = D - A$ and $Q(G) = D + A$. We write $\lambda_1, \ldots, \lambda_n$ and $q_1, \ldots, q_n$ for the eigenvalues of A and $Q(G)$ indexed in descending order, and $\mu_1, \ldots, \mu_n$ for the eigenvalues of $L(G)$ indexed in ascending order. Occasionally we write $q_{\min}$ and $\lambda_{\min}$ for q_n and λ_n. For more details on the Q matrix, we refer the reader to [6].

Here is our new problem:

Problem A. Let $n > r \geq 2$. How large can $q_n(G)$ be if G is graph of order n with no complete subgraph of order $r+1$?

* Corresponding author.
E-mail addresses: leonardo.lima@cefet-rj.br (L. de Lima), vnikifrv@memphis.edu (V. Nikiforov), carla.oliveira@ibge.gov.br (C. Oliveira).

http://dx.doi.org/10.1016/j.disc.2016.02.002
0012-365X/

Note that Problem A is in the spirit of the classical extremal graph theory, where the analog of Problem A is answered by the Turán theorem. To state this theorem, let $e(G)$ denote the number of edges of G, write K_r for the complete graph of order r, and write $T_r(n)$ for the complete r-partite graph of order n, with parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$.

Theorem T (*Turán, [17]*)**.** *If $n > r \geq 2$ and G is a K_{r+1}-free graph of order n, then $e(G) < e(T_r(n))$, unless $G = T_r(n)$.*

Problem A seems a like of Theorem T, but this similarity is superficial, for it turns out that Problem A is a much deeper question, entangled with a notoriously difficult open problem in extremal graph theory. To substantiate this claim, let us state a theorem, which at first glance seems off-topic.

Theorem 1. *If G is a graph of order n, then one has to remove at least $q_n n/4$ edges to make G bipartite.*

We shall prove Theorem 1 in Section 2.1, but let us mention that Brandt [4] has already proved the same assertion for regular graphs, by a different method and with a different terminology. However, our Theorem 1 turns out to be much more useful. The reason is that the topic of making graphs bipartite has been studied for longtime, with several usable results, which in view of Theorem 1 directly apply to Problem A.

Let us recall that in [9] Erdős initiated the study of the problem: *How many edges must be deleted from a given graph to make it bipartite?* In particular, he made the following conjecture:

Conjecture 2. *Every triangle-free graph of order n can be made bipartite by removing at most $n^2/25$ edges.*

Defying 46 years of attacks, Conjecture 2 is still widely open. Nonetheless, a few nontrivial results are known (see, e.g., [4,10,11,16]), which we shall use below for partial answers to Problem A.

Conjecture 2 can be extended for K_r-free graphs; for example, in [10] it was conjectured that a K_4-free graph can be made bipartite by deleting at most $n^2/9$ edges. This conjecture was fully proved by Sudakov in [16]—one of the few definite results in this area. In Section 1.3 we shall use Sudakov's result to get a corollary about $q_{\min}$ of K_4-free graphs.

However, the progress with Problem A along this line can go only so far, and it is unlikely that it can be reduced to a question about making a graph bipartite. Indeed, Problem A seems to have its own level of difficulty and its solution may take a while.

To get started, one can simplify Problem A by restating it for regular graphs:

Problem B. Let $n > r \geq 2$. How large can $q_n(G)$ be if G is a regular K_{r+1}-free graph of order n?

This step is well justified, for first, the known upper bounds can be considerably reduced for regular graphs, and second, it is likely that the extremal graphs in Problem A are regular or close to regular. Hence, Problem B may provide useful intuition for Problem A.

Moreover, Brandt [4] has already obtained several results for $q_{\min}$ of regular K_{r+1}-free graphs, albeit stated in different terms. We shall recall some of these results in due course below.

1.1. The function $f_r(n)$ and its asymptotics

To study Problem A in a systematic way let us define the function

$$f_r(n) := \max \{q_n(G) : G \text{ is a graph of order } n \text{ and } G \text{ contains no } K_{r+1}\}.$$

With $f_r(n)$ in hand, we can give a more formal statement of Problem A:

Problem 3. For any $r \geq 2$ and $n > r$, find or estimate $f_r(n)$.

Note that the introduction of $f_r(n)$ does not advance the solution of Problem A in any concrete way, yet it allows to clearly see and track the two main lines of attack: on the one hand, obtaining upper bounds on $f_r(n)$ by proofs, and on the other hand, obtaining lower bounds on $f_r(n)$ by constructions. The ultimate goal is to close the gap between the upper and lower bounds, which might take some time.

Before presenting concrete bounds we shall come up with general asymptotics of $f_r(n)$. For every $r \geq 2$, let us define the real number c_r as

$$c_r := \sup \{q_{\min}(G)/v(G) : G \text{ is a graph with no } K_{r+1}\}.$$

Because $q_n(G) \leq v(G) - 2$ [18], we see that c_r is well defined. Clearly, the definition of c_r implies a simple universal bound for any K_{r+1}-free graph G of order n:

$$q_n(G) \leq c_r n.$$

What is more, this bound is asymptotically best possible, as given by the next theorem: