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The vertex size-Ramsey number

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we study an analogue of size-Ramsey numbers for vertex colorings. For a given number of colors r and a graph G the vertex size-Ramsey number of G, denoted by $\hat{R}_v(G, r)$, is the least number of edges in a graph H with the property that any r-coloring of the vertices of H yields a monochromatic copy of G. We observe that $\Omega_r(\Delta n) = \hat{R}_v(G, r) = O_r(n^2)$ for any G of order n and maximum degree Δ, and prove that for some graphs these bounds are tight. On the other hand, we show that even 3-regular graphs can have nonlinear vertex size-Ramsey numbers. Finally, we prove that $\hat{R}_v(T, r) = O_r(\Delta^2 n)$ for any tree of order n and maximum degree Δ, which is only off by a factor of Δ from the best possible.

1. Introduction

Ramsey's theorem has inspired many beautiful and difficult problems (see, e.g., [13]). The study of the size-Ramsey numbers, introduced in the late 1970s by Erdős, Faudree, Rousseau and Schelp [10], is one of them. We write $H \to (G)^e_r$ if any r-coloring of the edges of H yields a monochromatic copy of G. The *(edge) size-Ramsey number* of G is defined as

$$\hat{R}_e(G, r) = \min\{|E(H)| : H \to (G)^e_r\}.$$

The size-Ramsey numbers have attracted a lot of attention and have been studied by several researchers (see, e.g., [11]). In this paper we consider an analogous number for vertices. We write $H \to (G)^v_r$ if any r-coloring of *vertices* of H yields a monochromatic copy of G and define the *vertex size-Ramsey number* as

$$\hat{R}_v(G, r) = \min\{|E(H)| : H \to (G)^v_r\}.$$

It is not difficult to see that

$$\hat{R}_v(G, r) \le \hat{R}_e(G, r(r+1)/2) \tag{1}$$

provided that G contains an odd cycle and has no isolated vertices. To see this, suppose that we have a graph H with $|E(H)| = \hat{R}_e(G, r(r+1)/2)$ such that $H \to (G)^e_{r(r+1)/2}$. For an arbitrary r-coloring of the vertices of H, we color the edges of H with colors depending on the endpoints, namely, we assign to each edge an unordered pair consisting of the colors of its endpoints. This way we obtain an edge coloring with at most $\binom{r}{2} + r = r(r+1)/2$ colors. Now we find a monochromatic copy of G in this edge-coloring. But this must be monochromatic in the original vertex coloring, since G contains an odd

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cycle. The next observation shows that inequality (1) is only useful for graphs G of order n for which $\hat{R}_e(G, r(r+1)/2))$ is at most quadratic in n. Otherwise, one trivially obtains a better bound by noting that

$$\hat{R}_v(G, r) \le \binom{r(n-1)+1}{2}, \tag{2}$$

since $K_{r(n-1)+1} \to (K_n)^v_r$ and $G \subseteq K_n$.

Now we establish some general lower bounds. (In the edge-coloring case such lower bounds are unknown.)

Proposition 1.1. *Let G be a graph of order n with maximum degree Δ. Let $n_\Delta = |\{v \in V(G) : \deg(v) = \Delta\}|$. Then,*

$$\hat{R}_v(G, r) \ge \frac{((r-1)(n-1)+n_\Delta)\Delta}{2}.$$

This immediately implies:

Corollary 1.2. *Let G be a graph of order n with maximum degree Δ. Then,*

$$\hat{R}_v(G, r) \ge \frac{((r-1)(n-1)+1)\Delta}{2}.$$

For d-regular graphs we have a more accurate bound.

Proposition 1.3. *Let G be a d-regular graph of order n. Then,*

$$\hat{R}_v(G, r) \ge \frac{(r(n-1)+1)dr}{2}.$$

This bound in general cannot be improved since, for example, together with (2) it yields:

Corollary 1.4. *For any positive integers n and r,*

$$\hat{R}_v(K_n, r) = \binom{r(n-1)+1}{2}.$$

(One can also prove this equality by adapting an argument from a paper of Burr, Erdős and Lovász [6].) This corollary shows that the vertex size-Ramsey number of a complete graph is exactly what one can expect from the trivial upper bound (2) and can be also viewed as an analogue of a theorem of Chvátal (Theorem 1 in [10]), who proved that $\hat{R}_e(K_n, 2) = \binom{R(n)}{2}$, where $R(n)$ is the classical Ramsey number. On the other hand, for some d-regular graphs G, Proposition 1.3 gives bounds which are asymptotically smaller than $\hat{R}_v(G, r)$ (cf. Theorem 1.10).

Summarizing the above results in asymptotic notation, we obtain that for every graph G of order n with maximum degree Δ,

$$\Omega_r(\Delta n) = \hat{R}_v(G, r) = O_r(n^2), \tag{3}$$

where subscript r indicates that the hidden constants depend only on r. For example, for any G of order n with $\Delta(G) = \Omega_r(n)$ we have $\hat{R}_v(G, r) = \Theta_r(n^2)$. In particular, $\hat{R}_v(K_{1,n-1}, r) = \Theta_r(n^2)$ which significantly differs from the edge case in which $\hat{R}_e(K_{1,n-1}, r) = \Theta_r(n)$. For the latter it is enough to observe that $K_{1,r(n-2)+1} \to (K_{1,n-1})^e_r$. Moreover, we show that the lower bound $\hat{R}_v(G, r) = \Omega_r(\Delta n)$ is tight for every Δ. It follows from a result of Friedman and Pippenger [12] that this is true for any tree of bounded constant degree.

Theorem 1.5 ([12]). *Let $\varepsilon > 0$ and Δ be fixed constants. Then there exists a graph $G = (V, E)$ such that $|E| = O(n)$ and for any subset $U \subseteq V$ with $|U| \ge \varepsilon|V|$ the graph $G[U]$ contains all trees of order n and maximum degree at most Δ.*

Applying this theorem with $\varepsilon = 1/r$ implies that $\hat{R}_v(T, r) = O_r(n)$ for any tree T of order n and bounded constant degree. We show that the lower bound in (3) also holds for a class of trees with an arbitrarily large maximum degree.

We say that a tree T of order n has the *s-separation property* if there exists a sequence of trees $T_1 \subsetneq T_2 \subsetneq \ldots \subsetneq T_n = T$ such that for every $1 \le i \le n-1$:

(i) T_{i+1} is obtained from T_i by attaching a new leaf to T_i, and
(ii) for $\{\{u, v\}\} = E(T_{i+1}) \setminus E(T_i)$ with $u \in V(T_{i+1})$ (i.e., u was attached to $v \in V(T_i)$), $v \notin V(T_{i-s})$. (This is vacuously true if $s \ge i$.)

It is not difficult to see that if T has the s-separation property, then the bandwidth of T (see, e.g., [7]) is at most s.

Theorem 1.6. *Let $r \ge 2$ be fixed. Then for each tree T of order n with the s-separation property we have*

$$\hat{R}_v(T, r) = O_r(sn).$$