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# On the completability of incomplete orthogonal Latin rectangles

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ABSTRACT

We address the problem of completability for 2-row orthogonal Latin rectangles (*OLR*2). Our approach is to identify all pairs of incomplete 2-row Latin rectangles that are not completable to an *OLR*2 and are minimal with respect to this property; i.e., we characterize all circuits of the independence system associated with *OLR*2. Since there can be no polytime algorithm generating the clutter of circuits of an arbitrary independence system, our work adds to the few independence systems for which that clutter is fully described. The result has a direct polyhedral implication; it gives rise to inequalities that are valid for the polytope associated with orthogonal Latin squares and thus planar multi-dimensional assignment. A complexity result is also at hand: completing a set of $(n-1)$ incomplete *MOLR*2 is $\mathcal{NP}$-complete.



## 1. Introduction

An $m$-row Latin rectangle $R$ of order $n$ is an $m \times n$ array where $m < n$, in which each value $1, \ldots, n$ appears exactly once in every row and at most once in every column [12]. For $m = n$, the above defines a *Latin square*, where each value $1, \ldots, n$ appears exactly once in every row and column. We call a Latin rectangle *normalized* if values $1, \ldots, n$ occur in the first row in natural order. Counting Latin rectangles is a topic broadly studied in combinatorics; some examples listed in chronological order are [16,4,7,13] and [18].

**Definition 1.** Two $m$-row Latin rectangles of order $n$, with $m < n$, form an orthogonal pair (OLR) if and only if when superimposed each of the $n^2$ ordered pairs of values $(1, 1), (1, 2), \ldots, (n, n)$ appears at most once.

An example of a normalized 2-row OLR (*OLR*2) of order 4 is shown in Table 1. Also note that for $m = n$ we have the case of *orthogonal Latin squares (OLS)* where each of the $n^2$ ordered pairs of values $(1, 1), (1, 2), \ldots, (n, n)$ appears exactly once when the two squares are superimposed.

The definition for OLR naturally extends to a set $T$ of $m$-row Latin rectangles of order $n$, which are called *mutually orthogonal Latin rectangles (MOLR)*, if and only if all Latin rectangles are pairwise orthogonal. Note that for $m = n$ we have the case of mutually orthogonal Latin squares (MOLS). Here, we are interested only in 2-row Latin rectangles (i.e., *MOLR*2). Hence, unless otherwise stated, whenever we refer to Latin rectangles we imply that they have two rows.

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**Table 1**
An *OLR*2 of order 4.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 1 | | 3 | 4 | 1 | 2 |

Latin rectangles and OLRs enjoy a close relationship to several areas of combinatorics like design theory and projective geometry (e.g., see [12] and references therein). Beyond that, they have recently received additional attention because of some quite important applications:

- [15] and [22] introduce the concept of physical layer network coding which has developed into a sub-field of network coding with new results in the domains of wireless communication, wireless information theory and wireless networking. One branch of this new field works with de-noise-and-forward-protocol in the network coding maps that satisfy a requirement called the 'exclusive law', which reduces the impact of multiple access interference. In [21] it is established that the network coding maps that satisfy the 'exclusive law' are obtainable by the completion of incomplete Latin rectangles. Isotopic and transposed Latin squares are also used to create network coding maps with particular desirable characteristics.
- Fibre-optic signal processing techniques [17] deliver multi-access optical networks for fibre-optic communications. Relevant to that, an Optical Orthogonal Code (OOC) is a family of (0, 1) sequences with good auto and cross-correlation properties, i.e., fast and low interference transmission properties. In [3] the authors propose two new coding schemes capable of cancelling the multi-user interference for certain systems based on MOLR and MOLS to accomplish large flexibility in choosing number of users, simplicity of construction and suitability to all important transmission technologies.
- LDPC codes are the lead technology used in hard disk drive read channels, wireless 10-GB, DVB-S2 and more recently in flash SSD as well as in communicating with space probes. Pseudo-random approaches and combinatorial approaches are the two main techniques for the construction of a specific LDPC code, based on finite geometries and first studied in [9]. In [20] and [10] a different construction is devised, based on balanced incomplete block designs constructed from *MOLR* and *MOLS*.

In this paper, after establishing that *MOLR* completion is $\mathcal{NP}$-complete, we address the problem of completability for *OLR*2. To achieve this, it suffices to characterize all pairs of incomplete 2-row Latin rectangles that are not completable to an *OLR*2. Minimal such pairs define *circuits* of the *independence system* (*IS*) associated with *OLR*2 of order $n$ (formal definitions appear in the next section). In this system, a pair of incomplete 2-row Latin rectangles is independent if and only if it is completable to an *OLR*2 or, equivalently, it contains no circuits. Notably, the circuits for the IS associated with 2-row Latin rectangles have been described in [5]; this description has been based on the notion of an availability matrix, which is also employed here to provide a concise proof, despite the enumerative nature of our exposition.

Since there can be no polytime algorithm (unless $\mathcal{P} = \mathcal{NP}$) generating the clutter of either bases or circuits of an arbitrary IS [19], our work adds to the (few) independence systems in the literature for which the clutter of circuits is fully characterized (see [6], [19] and [11]). The results presented here have some polyhedral implications that are also discussed, namely they directly give rise to *lifted circuit inequalities* for the polytope associated with both *OLR* and *OLS*. Finally, our approach could be useful for the characterization of both circuits and associated inequalities for *MOLR*, *MOLS* and possibly other highly symmetric combinatorial problems.

The remainder of this paper is organized as follows. In Section 2, we introduce our notation and present some initial results, including the complexity of *MOLR* completion. After reviewing the results of [5], along with counting the circuits associated with the completion of 2-row (single) Latin rectangles in Section 3, we present our main contribution in Section 4. In Section 5, we discuss the implications of our work regarding the polytope associated with orthogonal Latin squares and thus planar multi-dimensional assignment [1]. We conclude in Section 6 with ideas for future work. The proofs of some intermediate results appear in the Appendix.

## 2. Notation and basic results

Let us introduce our notation. For a given order $n$, let $T = \{1, \ldots, |T|\}$, where $|T| \leq (n-1)$ is the number of *MOLR*2. The two sets $I = \{i_1, i_2\}$ and $J = \{j_1, \ldots, j_n\}$ correspond to the rows and columns of each Latin rectangle, while the $|T|$ disjoint sets $K_t = \{k_1^t, \ldots, k_n^t\}$ $(t \in T)$ define the $n$ elements appearing in the $t$th *MOLR*2. Define $G_t = I \times J \times K_t$, $t \in T$, i.e., each $G_t$ contains $2n^2$ triples and $\bigcup_{t \in T} G_t$ contains $2|T|\, n^2$ triples. Based on this notation, Table 1 is revised in Table 2, whereas a Latin rectangle can be represented as an $R_t \subseteq G_t$, e.g., $R_1$ in Table 2 can be written as

$$R_1 = \{(i_1, j_1, k_1^1), (i_1, j_2, k_2^1), (i_1, j_3, k_3^1), (i_1, j_4, k_4^1), (i_2, j_1, k_2^1), (i_2, j_2, k_3^1), (i_2, j_3, k_4^1), (i_2, j_4, k_1^1)\}.$$

Similarly, an *OLR*2 is represented as $R_t \cup R_{t'}$, where $R_t \subseteq G_t$, $R_{t'} \subseteq G_{t'}$ and $\{t, t'\} \subseteq T$; for example the *OLR*2 of Table 2 is represented as $R_1 \cup R_2$.

We call two *MOLR*2 *equivalent*, if one is obtainable from the other by permuting the sets $I, J, K_1, \ldots, K_{|T|}$ and $T$. Note that directly interchanging the roles of sets $I$ and $J$ is not allowed since we stipulate the existence of only 2 rows but $n$ columns;