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On forbidden subgraphs and rainbow connection in graphs with minimum degree 2

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ABSTRACT

A connected edge-colored graph G is said to be rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors, and the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors that can make G rainbow-connected. We consider families $\mathcal{F}$ of connected graphs for which there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G with minimum degree at least 2 satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $\mathrm{diam}(G)$ is the diameter of G. In this paper, we give a complete answer for $|\mathcal{F}| = 1$, and a partial answer for $|\mathcal{F}| = 2$.

1. Introduction

We consider undirected finite simple graphs, and for terminology and notation not defined here we refer to [3]. To avoid trivial cases, all graphs considered here will be connected with at least one edge.

An edge-colored connected graph G is called *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, that is, by a path whose edges have pairwise distinct colors. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors that can make G rainbow-connected.

The concept of rainbow connection was introduced by Chartrand et al. in [7]. It is easy to observe that if G has n vertices then $\mathrm{rc}(G) \leq n-1$, since we can color the edges of some spanning tree of G with different colors and then color the remaining edges with one of the already used colors. Chartrand et al. determined the precise value of the rainbow connection number for several graph classes including complete multipartite graphs [7]. The rainbow connection number has been studied for further graph classes in [4,8,11,14] and for graphs with fixed minimum degree in [4,6,12,16]. See [15] for a survey.

The computation of $\mathrm{rc}(G)$ is known to be NP-hard [5,13]. In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$, and it is also NP-complete to decide whether a given edge-colored graph (with an unbounded number of colors) is rainbow-connected [5]. More generally, it has been shown in [13] that for any fixed $k \geq 2$ it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

In the following proposition, we summarize some obvious facts and observations for the rainbow connection number of graphs.

Proposition A. *Let G be a connected graph of order n. Then*

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(i) $1 \le \mathrm{rc}(G) \le n-1$,
(ii) $\mathrm{rc}(G) \ge \mathrm{diam}(G)$,
(iii) $\mathrm{rc}(G) = 1$ *if and only if* G *is complete,*
(iv) $\mathrm{rc}(G) = n-1$ *if and only if* G *is a tree.*

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large, as can be seen by considering $G \simeq K_{1,n-1}$, for which $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n-1) - 2 = n-3$. Especially, each bridge of G requires a single color. Therefore, connected bridgeless graphs have been studied.

Theorem B *([2]). For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \le r(r+2).$$

Moreover, for every integer $r \ge 1$, there exists a bridgeless graph G with radius r and $\mathrm{rc}(G) = r(r+2)$.

Note that, since $\mathrm{rad}(G) \le \mathrm{diam}(G)$, Theorem B gives in bridgeless graphs an upper bound on $\mathrm{rc}(G)$ which is quadratic in terms of the diameter of G. In this paper, we will be interested in finding conditions on a graph G that imply a linear upper bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is *X-free*, and for $\mathcal{F} = \{X, Y\}$ we say that G is *(X, Y)-free*. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*, and for $|\mathcal{F}| = 2$ we also say that $\mathcal{F}$ is a *forbidden pair*.

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, by virtue of Theorem B, $\mathrm{rc}(G)$ can be (even for bridgeless graphs) still quadratic in terms of $\mathrm{diam}(G)$, it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\mathrm{rc}(G)$.

In [10], the authors considered the question for which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$), and gave a complete answer for $1 \le |\mathcal{F}| \le 2$ by the following two results (where N denotes the *net*, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem C *([10]). Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_X$, if and only if $X = P_3$.*

Theorem D *([10]). Let X, Y be connected graphs, $X, Y \ne P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$, $r \ge 4$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.*

Moreover, it was also shown in [10] that the (seemingly more general) question of finding families $\mathcal{F}$, $1 \le |\mathcal{F}| \le 2$, implying a linear upper bound on $\mathrm{rc}(G)$, i.e., such that every connected $\mathcal{F}$-free graph G satisfies $\mathrm{rc}(G) \le q_{XY} \cdot \mathrm{diam}(G) + k_{XY}$, where q_{XY}, k_{XY} are constants, has the same solution as in Theorems C, D.

In this paper, we will consider an analogous question under an additional assumption $\delta(G) \ge 2$. Under this assumption, such an upper bound on $\mathrm{rc}(G)$ is already known for graphs from some special classes of graphs, such as e.g. interval graphs, AT-free graphs, threshold graphs or circular arc graphs (see [6], or Theorem 5.2.2 in [15]). In this paper, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G with $\delta(G) \ge 2$ being $\mathcal{F}$-free implies $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{\mathcal{F}}$?

We give a complete answer for $|\mathcal{F}| = 1$ in Section 3, and a partial answer for $|\mathcal{F}| = 2$ in Section 4. Finally, in Section 5 we show that there are no more families with $|\mathcal{F}| \le 2$ that would imply a linear bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$ for connected graphs G with $\delta(G) \ge 2$.

2. Preliminary results

In this section we summarize some further notations and facts that will be needed for the proofs of our results.

An edge $e \in E(G)$ such that $G - e$ is disconnected is called a *bridge*, and a graph with no bridges is called a *bridgeless graph*. An edge such that one of its vertices has degree one is called a *pendant edge*. The *subdivision* of a graph G is the graph obtained from G by adding a vertex of degree 2 to each edge of G. For graphs X, G, we write $X \subset G$ if X is a subgraph of G, $X \overset{\mathrm{IND}}{\subset} G$ if X is an induced subgraph of G, and $X \simeq G$ if X and G are isomorphic. For two vertices $x, y \in V(G)$, we denote by $\mathrm{dist}(x, y)$ the distance between x and y in G. The diameter and the radius of a graph G will be denoted by $\mathrm{diam}(G)$ and $\mathrm{rad}(G)$, respectively. A shortest path joining two vertices at distance $\mathrm{diam}(G)$ will be referred to as a *diameter path*.

For a set $S \subset V(G)$ and an integer $k \ge 1$, the *neighborhood at distance k* of S is the set $N_G^k(S)$ of all vertices of G at distance k from S. In the special case when $k = 1$, we simply write $N_G(S)$ for $N_G^1(S)$, and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subset V(G)$, we set $N_M(S) = N_G(S) \cap M$ and $N_M(x) = N_G(x) \cap M$, and for a subgraph $P \subset G$, we write $N_P(x)$