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Fool's solitaire on joins and Cartesian products of graphs

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ABSTRACT

Peg solitaire is a game generalized to connected graphs by Beeler and Hoilman. In the game pegs are placed on all but one vertex. If xyz form a 3-vertex path and x and y each has a peg but z does not, then we can remove the pegs at x and y and place a peg at z. By analogy with the moves in the original game, this is called a *jump*. The goal of the peg solitaire game on graphs is to find jumps that reduce the number of pegs on the graph to 1.

Beeler and Rodriguez proposed a variant where we instead want to maximize the number of pegs remaining when no more jumps can be made. Maximizing over all initial locations of a single hole, the maximum number of pegs left on a graph G when no jumps remain is the fool's solitaire number $F(G)$. We determine the fool's solitaire number for the join of any graphs G and H. For the Cartesian product, we determine $F(G \square K_k)$ when $k \geq 3$ and G is connected and show why our argument fails when $k = 2$. Finally, we give conditions on graphs G and H that imply $F(G \square H) \geq F(G)F(H)$.

1. Introduction

Peg solitaire is a game generalized to connected graphs by Beeler and Hoilman [4]. In the peg solitaire game on graphs, each vertex except one starts with a peg. Vertices without pegs are said to be *holes*. If adjacent vertices x and y have pegs, and z adjacent to y is a hole, then we may *jump* the peg at x over the peg at y and into the hole at z. This removes the peg at y so that x and y become holes and z has a peg. We denote this jump by xyz.

In general, if we start with some configuration of pegs and holes, and some succession of jumps reduces the number of pegs to 1, then the configuration is *solvable*. In the peg solitaire game on a graph G, if some configuration with a hole at one vertex and pegs at all other vertices is solvable, then we say G is *solvable*. If G can be solved starting with a single hole at any vertex, then G is *freely solvable*. Solvability requires G to be connected.[1]

Beeler and Hoilman [4] determined which graphs are solvable and freely solvable among stars, paths, cycles, complete graphs, and complete bipartite graphs. They also proved that the Cartesian products of solvable graphs are solvable and gave additional sufficient conditions for the solvability of Cartesian products of graphs. Walvoort [1] also determined which of the trees of diameter 4 are solvable.

An alternate goal for the peg solitaire game was proposed in [6]. In the *fool's solitaire* game, we instead try to maximize the number of pegs at the end of the process (when there are no remaining available moves). A *terminal state* is a set of vertices that are the final locations of pegs when the game is played starting with a single hole, and played until no more jumps are

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[1] There are several traditional boards marketed commercially, a triangle with 15 positions in the US, a portion of a grid in England (marketed as "Hi-Q" in the US), and a European board with more positions than the English board. The significant distinction between these games and the graph version is that they restrict jumps to be made along geometric straight lines.

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possible. Because no more jumps are possible from a terminal state, all terminal states are independent sets of vertices. The *fool's solitaire number* of a graph G is the maximum size of a terminal state and is denoted by $F(G)$. A fundamental observation follows from the fact that moves from a configuration are the reverse of moves from the complementary configuration.

Proposition 1 ([6]). *A set of vertices T is a terminal state of some solitaire game on G if and only if a starting configuration with holes at vertices of T and pegs at vertices of $V(G) - T$ can be reduced to a single peg.*

Proposition 1 is used in our proofs of lower bounds on the fool's solitaire number. Letting $\alpha(G)$ denote the independence number of G, Beeler and Rodriguez [6] also proved.

Proposition 2 ([6]). *Let G be a graph. Because terminal states are independent sets, $F(G) \leq \alpha(G)$. Also, if $\alpha(G) \leq |V(G)| - 2$ and $V(G) - A$ is independent whenever A is a maximum independent set, then $F(G) \leq \alpha(G) - 1$.*

The proposition holds since if the complement of every maximum independent set is independent and has at least two vertices, then by Proposition 1 no maximum independent set can be the terminal state of a solitaire game.

The fool's solitaire numbers for complete graphs, stars, complete bipartite graphs, paths, cycles, and hypercubes were found in [6]. The fool's solitaire number of trees with diameter 4 was computed by Walvoort [1]. In particular, there is a class of diameter 4 trees for which $\alpha(G) - F(G)$ approaches $\alpha(G)/6$, disproving an earlier conjecture that $\alpha(G) - F(G) \leq 1$ [6]. It remains open how small $F(G)$ can be in terms of $\alpha(G)$.

Beeler and Rodriguez [6] proved $F(K_{n,m}) = \alpha(K_{n,m}) - 1$, and thus Proposition 2 is sharp. In Section 2, we extend their result on complete bipartite graphs by determining the fool's solitaire number of all graphs whose complements are disconnected.

Beeler and Rodriguez [6] also asked for the behavior of the fool's solitaire number under the Cartesian product operation. The *Cartesian product* of G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. In Section 3, we show $F(G \square K_k) = \alpha(G \square K_k)$ for $k \geq 3$ when G is any connected graph. However, this behavior does not hold when $k = 2$: if G is a bipartite graph with a Hamiltonian path, then $F(G \square K_2) = \alpha(G \square K_2) - 1$. This leads us to ask:

Question 1. What is $F(G \square K_2)$ when G is not a bipartite graph having a Hamiltonian path?

Walvoort [1] asked for a non-trivial lower bound on $F(G)$. In this direction, we give sufficient conditions for $F(G \square H) \geq F(G)F(H)$ in Section 4. This is a partial answer to the question in [6] asking for the relationship among $F(G)$, $F(H)$, and $F(G \square H)$. In considering the sharpness of our inequality, we ask:

Question 2. By how much can $F(G \square H)$ exceed $F(G)F(H)$?

Computer testing shows that $F(G \square H) \geq F(G)F(H)$ does not always hold: if G is the star with 4 vertices and H is the paw graph or the path on three vertices (P_3), then $F(G \square H) = F(G)F(H) - 1$. This leads to the question:

Question 3. When does $F(G)F(H)$ exceed $F(G \square H)$?

In another direction, in Lemma 5 we show that complete graphs with more than four vertices have the property that one can begin with a hole at any vertex and can solve the graph so that the final peg is at any specified vertex. This raises a question about a restriction on the idea of freely solvable graphs.

Question 4. What graphs, other than K_k for $k > 4$, can start with one hole in any specified vertex and end with one peg at any specified vertex?

Bell [3] determined that several geometrically defined boards have this property.

2. Joins

The *join* of G and H, denoted by $G \oplus H$, is formed by adding to the disjoint union of G and H all edges joining $V(G)$ and $V(H)$. Note that every join is connected and these are precisely the graphs whose complements are disconnected. For the complete bipartite graph $K_{n,m}$ with $n \geq m > 1$, Beeler and Rodriguez [6] showed $F(K_{n,m}) = n - 1$. By viewing $K_{n,m}$ as $\overline{K_n} \oplus \overline{K_m}$, we expand their method to find the fool's solitaire number of all graph joins, starting with the case of joins with K_1.

Lemma 3. *If G is a graph, then $F(G \oplus K_1) = \alpha(G \oplus K_1)$.*

Proof. Always $F(G \oplus K_1) \leq \alpha(G \oplus K_1) = \alpha(G)$, so we must show $F(G \oplus K_1) \geq \alpha(G \oplus K_1)$. If $G = \overline{K_n}$, then $G \oplus K_1$ is a star and $F(G \oplus K_1) = \alpha(G \oplus K_1)$ because there are no available moves if we place the starting hole at the center of the star. Otherwise, let S be a largest independent set of G, and let z be the vertex outside G. We wish to show that S is a terminal state; by Proposition 1 it suffices to solve the game where S gives the locations of the starting holes. Since S is a maximum independent set, every peg is adjacent to a hole in G. Start by jumping any peg in G over the peg at z, and landing in a hole adjacent to another peg in G. We now have two adjacent pegs and we next jump one over the other and land at the hole at z. By repeating this two-jump process the number of pegs is reduced to 1.

The remaining case is when $G \oplus H$ is not a complete bipartite graph and has no dominating vertex.

Theorem 4. *Let G and H be graphs with $|V(G)|, |V(H)| \geq 2$ and $|E(G)| + |E(H)| \geq 1$. Then $F(G \oplus H) = \alpha(G \oplus H)$.*