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Weighted well-covered claw-free graphs

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ABSTRACT

A graph G is *well-covered* if all its maximal independent sets are of the same cardinality. Assume that a weight function w is defined on its vertices. Then G is w-*well-covered* if all maximal independent sets are of the same weight. For every graph G, the set of weight functions w such that G is w-well-covered is a *vector space*. Given an input claw-free graph G, we present an $O(m^{\frac{3}{2}}n^3)$ algorithm, whose input is a claw-free graph G, and output is the vector space of weight functions w, for which G is w-well-covered.

A graph G is *equimatchable* if all its maximal matchings are of the same cardinality. Assume that a weight function w is defined on the edges of G. Then G is w-*equimatchable* if all its maximal matchings are of the same weight. For every graph G, the set of weight functions w such that G is w-equimatchable is a vector space. We present an $O\left(m \cdot n^4 + n^5 \log n\right)$ algorithm, which receives an input graph G, and outputs the vector space of weight functions w such that G is w-equimatchable.

1. Introduction

1.1. Basic definitions and notation

Throughout this paper G is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V(G)$ and edge set $E(G)$. In what follows, we denote $n = |V(G)|$ and $m = |E(G)|$.

Cycles of k vertices are denoted by C_k, and paths of k vertices are denoted by P_k. When we say that G contains a C_k or a P_k for some $k \geq 3$, we mean that G admits a subgraph isomorphic to C_k or to P_k, respectively. It is important to mention that these subgraphs are not necessarily induced.

Let u and v be two vertices in G. The *distance* between u and v, denoted as $d(u, v)$, is the length of a shortest path between u and v, where the length of a path is the number of its edges. If S is a non-empty set of vertices, then the *distance* between u and S, denoted as $d(u, S)$, is defined by

$$d(u, S) = \min\{d(u, s) : s \in S\}.$$

For every positive integer i, denote

$$N_i(S) = \{x \in V(G) : d(x, S) = i\},$$

and

$$N_i[S] = \{x \in V(G) : d(x, S) \leq i\}.$$

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We abbreviate $N_1(S)$ and $N_1[S]$ to be $N(S)$ and $N[S]$, respectively. If S contains a single vertex, v, then we abbreviate

$$N_i(\{v\}), N_i[\{v\}], N(\{v\}), \text{ and } N[\{v\}]$$

to be

$$N_i(v), N_i[v], N(v), \text{ and } N[v],$$

respectively. We denote by $G[S]$ the subgraph of G induced by S. For every two sets, S and T, of the vertices of G, we say that S *dominates* T if $T \subseteq N[S]$.

1.2. Well-covered graphs

Let G be a graph. A set of vertices S is *independent* if its elements are pairwise nonadjacent. An independent set of vertices is *maximal* if it is not a subset of another independent set. An independent set of vertices is *maximum* if the graph does not contain an independent set of a higher cardinality.

The graph G is *well-covered* if every maximal independent set is maximum. Assume that a weight function $w : V(G) \longrightarrow \mathbb{R}$ is defined on the vertices of G. For every set $S \subseteq V(G)$, define

$$w(S) = \sum_{s \in S} w(s).$$

Then G is w-*well-covered* if all maximal independent sets of G are of the same weight.

The problem of finding a maximum independent set in an input graph is **NP**-complete. However, if the input is restricted to well-covered graphs, then a maximum independent set can be found polynomially using the *greedy algorithm*. Similarly, if a weight function $w : V(G) \longrightarrow \mathbb{R}$ is defined on the vertices of G, and G is w-well-covered, then finding a maximum weight independent set is a polynomial problem.

The recognition of well-covered graphs is known to be **co-NP**-complete. This was proved independently in [4,21]. In [3] it is proven that the problem remains **co-NP**-complete even when the input is restricted to $K_{1,4}$-free graphs. However, the problem is polynomially solvable for $K_{1,3}$-free graphs [22,23], for graphs with girth at least 5 [8], for graphs with a bounded maximal degree [2], for chordal graphs [19], for bipartite graphs [7,18,20], and for graphs without cycles of lengths 4 and 5 [9]. It should be emphasized that the forbidden cycles are not necessarily induced.

For every graph G, the set of weight functions w for which G is w-well-covered is a *vector space* [2]. That vector space is denoted as $WCW(G)$ [1].

Clearly, $w \in WCW(G)$ if and only if G is w-well-covered. Since recognizing well-covered graphs is **co-NP**-complete, finding the vector space $WCW(G)$ of an input graph G is **co-NP**-hard. In [16] there is a polynomial algorithm which receives as its input a graph G without cycles of lengths 4, 5, and 6, and outputs $WCW(G)$.

This article presents a polynomial algorithm whose input is a $K_{1,3}$-free graph G, and the output is $WCW(G)$. Thus we generalize [22,23], which supply a polynomial time algorithm for recognizing well-covered $K_{1,3}$-free graphs.

1.3. Generating subgraphs and relating edges

We use the following notion, which has been introduced in [14]. Let B be an induced complete bipartite subgraph of G on vertex sets of bipartition B_X and B_Y. Assume that there exists an independent set S such that each of $S \cup B_X$ and $S \cup B_Y$ is a maximal independent set of G. Then B is a *generating* subgraph of G, and it *produces* the restriction: $w(B_X) = w(B_Y)$. Every weight function w such that G is w-well-covered must *satisfy* the restriction $w(B_X) = w(B_Y)$. The set S is a *witness* that B is generating. In the restricted case that the generating subgraph B is isomorphic to $K_{1,1}$, call its vertices x and y. In that case xy is a *relating* edge, and $w(x) = w(y)$ for every weight function w such that G is w-well-covered.

The decision problem whether an edge in an input graph is relating is **NP**-complete [1]. Therefore, recognizing generating subgraphs is **NP**-complete as well. In [15] it is proved that recognizing relating edges and generating subgraphs is **NP**-complete even in graphs without cycles of lengths 4 and 5. However, recognizing relating edges can be done polynomially if the input is restricted to graphs without cycles of lengths 4 and 6 [13], and recognizing generating subgraphs is a polynomial problem when the input is restricted to graphs without cycles of lengths 4, 6 and 7 [14].

Generating subgraphs play an important roll in finding the vector space $WCW(G)$. In this article we use generating subgraphs in the algorithm which receives as its input a $K_{1,3}$-free graph G, and outputs $WCW(G)$.

1.4. Equimatchable graphs

Let G be a graph. The *line graph* of G, denoted as $L(G)$, is a graph such that every vertex of $L(G)$ represents an edge in G, and two vertices of $L(G)$ are adjacent if and only if they represent two edges in G with a common endpoint.

Every independent set of vertices in $L(G)$ defines a set of pairwise non-adjacent edges in G. A set of pairwise non-adjacent edges is called a *matching*. A matching M *saturates* a set S of vertices if every vertex of S is an endpoint of an edge of M. A matching in a graph is *maximal* if it is not contained in another matching.

The *size* of a matching M, denoted by $|M|$, is the number of its edges. A matching M is *maximum* if the graph does not admit a matching with size bigger than $|M|$.