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Non-matchable distributive lattices[☆]

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ABSTRACT

Based on an acyclic orientation of the Z-transformation graph, a finite distributive lattice (FDL for short) $\mathbf{M}(G)$ is established on the set of all 1-factors of a plane (weakly) elementary bipartite graph G. For an FDL $\mathbf{L}$, if there exists a plane bipartite graph G such that $\mathbf{L}$ is isomorphic to $\mathbf{M}(G)$, then $\mathbf{L}$ is called a *matchable FDL*. A natural question arises: Is every FDL a matchable FDL? In this paper we give a negative answer to this question. Further, we obtain a series of non-matchable FDLs by characterizing sub-structures of matchable FDLs with cut-elements.

1. Introduction

In this paper we take graph terminologies from [1]. As combinatorial structures, finite distributive lattices (FDLs for short) have been established on many combinatorial objects, such as the stable matchings of a bipartite graph [13], the set of flows of a planar graph [9] and the set of c-orientations with fixed flow difference on a plane graph [17,18] or in the dual setting [3]. A result of Propp [18] establishes some FDLs on the sets of d-factors, spanning trees, and Eulerian orientations in a plane (bipartite) graph.

The *Z-transformation graph* $Z(G)$ of a plane bipartite graph G having a 1-factor (that is, a perfect matching) is a simple graph on the set of all 1-factors of G: two 1-factors are adjacent if their symmetric difference is a cycle that is the boundary of a bounded face of G. This concept originates from Zhang et al. [25] for benzenoid graphs. In fact, this graph has been introduced independently several times under different names. For example, Gründler [6] introduced it, under the name *resonance graph*, on the set of Kekulé structures of benzenoid graphs. Randić [19] showed that the leading eigenvalue of the resonance graph correlates with the resonance energy of benzenoid by giving a quite satisfactory regression formula. Fournier [4] re-introduced this concept under the name *perfect matching graph* in domino tiling space. For more mathematical properties and chemical applications about Z-transformation graphs, the interested reader is referred to [14,33] and a recent survey [23] and references therein.

By distinguishing all alternating cycles with respect to some 1-factor of a plane bipartite graph into two classes, Zhang and Zhang [30] gave an orientation $\vec{Z}(G)$ on the Z-transformation graph $Z(G)$. The property that $\vec{Z}(G)$ is acyclic [31] yields a natural poset, denoted by $\mathbf{M}(G)$, on the set of 1-factors of G. In general, $\vec{Z}(G)$ is the Hasse diagram of $\mathbf{M}(G)$, and $Z(G)$ is the *cover graph* (also called undirected Hasse diagram) of $\mathbf{M}(G)$. For a plane (weakly) elementary bipartite graph G (its definition

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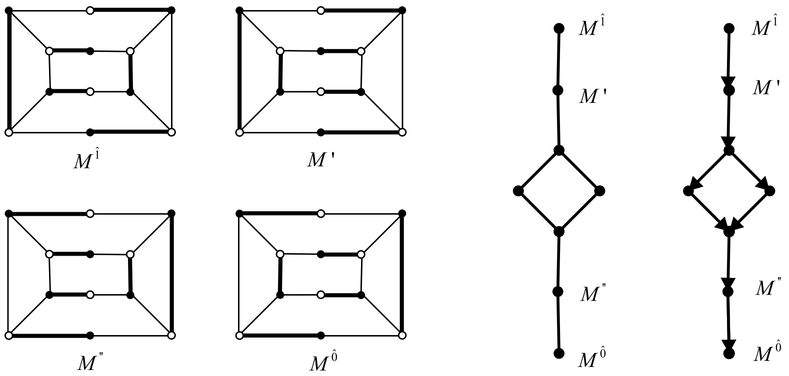

Fig. 1. A plane bipartite graph and its Z-transformation graph and digraph.

will be given in the next section), Lam and Zhang [14] proved that $\mathbf{M}(G)$ is an FDL. By applying such an FDL structure, Zhang et al. [26] gave three distance formulas for $\vec{Z}(G)$ and showed that the diameter of $Z(G)$ equals the height of $\mathbf{M}(G)$. Furthermore, they showed that every connected Z-transformation graph of a plane bipartite graph is a median graph. This extends the corresponding results of Klavžar et al. [12] on catacondensed benzenoid systems. For a plane bipartite graph that is not weakly elementary, Zhang [24] showed that $\mathbf{M}(G)$ is a direct sum of FDLs.

An FDL $\mathbf{L}$ is a *matchable FDL* [28] if there exists a plane bipartite graph G such that $\mathbf{L} \cong \mathbf{M}(G)$. So it is natural to ask whether every FDL is a matchable FDL. In fact, in this paper we will show that non-matchable FDLs exist. Thus, characterizing the matchable FDLs is a further problem. Zhang et al. [28] showed that for a plane elementary bipartite graph G, $\mathbf{M}(G)$ is irreducible. From this, a decomposition theorem is obtained: an FDL $\mathbf{L}$ is matchable if and only if for any direct product decomposition of $\mathbf{L}$, every factor is matchable.

The remainder of this paper consists of three sections. Some basic results about matchable FDLs are given in Section 2. In Section 3, we give some results and structures of Fibonacci cubes and Lucas cubes related to matchable FDLs. We show that some of these cubes can be and some cannot be the Z-transformation graphs of plane bipartite graphs. Thus sequences of matchable and non-matchable FDLs are obtained. In the last section, we construct a sequence of non-matchable FDLs by characterizing sub-structures of matchable FDLs with cut-elements, where a *cut-elements* of an FDL is a cut-vertex of its Hasse diagram, that is a cut-vertex of its cover graph, and a type (m, n) cut-element is a cut-element which is covered exactly by m elements and covers exactly n elements. By the planarity of the duals for plane graphs, we show that if a matchable FDL has a type (m, n) cut-element, then $\min\{m, n\} \leq 2$. We also show that a matchable FDL having a type $(2, n)$ cut-element with $n \geq 2$ must contain a special sublattice. Applying these results, we construct a sequence of non-matchable FDLs with cut-elements. By the decomposition theorem, some non-matchable FDLs without cut-elements are also obtained.

2. Matchable FDLs

Let G be a plane bipartite graph with a proper white/black coloring of its vertices, and let M be a 1-factor of G. A cycle C of G is *resonant* or *nice* if $G - V(C)$ has a 1-factor. A *cell* of G is a bounded face whose boundary is a cycle. In this paper we do not distinguish a cell from its boundary. We say that a face is *resonant* if its boundary is a resonant cycle. A bipartite graph is *elementary* [15] if it is connected and every edge belongs to a 1-factor. The complete graph K_2 with two vertices is the *trivial* elementary bipartite graph. Lovász and Plummer [15] showed that nontrivial elementary graphs are 2-connected. Also, a plane bipartite graph is a nontrivial elementary graph if and only if every face (including the outer face) is resonant (Zhang and Zhang [32]). A cycle or path of a graph is *M-alternating* if its edges are alternately in and out of M. Furthermore, an M-alternating cycle in a plane bipartite graph with 1-factor M is *proper* if under the clockwise orientation of the cycle the edges in M are oriented from white vertices to black vertices; otherwise it is *improper*.

The symmetric difference $A \oplus B$ of finite sets A and B is defined by $A \oplus B = (A \setminus B) \cup (B \setminus A)$. If A and B are subgraphs of a graph, then $A \oplus B$ is treated as the symmetric difference of their edge-sets.

Definition 2.1 (*[25,31]*). Fix a white/black proper coloring of a plane bipartite graph G having a 1-factor, and let $\mathcal{M}(G)$ be the set of all 1-factors of G. The Z-transformation digraph (or resonance digraph) of G, denoted by $\vec{Z}(G)$, is defined as the digraph on $\mathcal{M}(G)$ such that there exists an arc from M to M' if and only if the symmetric difference $M \oplus M'$ is a proper M-(thus improper M'-) alternating cell of G. Ignoring all directions of arcs of $\vec{Z}(G)$, we get the usual Z-transformation graph or resonance graph $Z(G)$ (see Fig. 1).

As noted in [31], the property that $\vec{Z}(G)$ is acyclic yields a partial ordering: for $M, M' \in \mathcal{M}(G)$, $M' \leq M$ if and only if $\vec{Z}(G)$ has a directed path from M to M'. As noted in [24], M covers M' if and only if $M' \oplus M$ is a proper M-(thus improper M'-) alternating cell. A change from M to M' on such a cell is a *twist* or *Z-transformation* on the cell. We denote this poset by $\mathbf{M}(G)$.