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Congruences for the number of partitions and bipartitions with distinct even parts

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ABSTRACT

Let $ped(n)$ denote the number of partitions of n wherein even parts are distinct (and odd parts are unrestricted). We show infinite families of congruences for $ped(n)$ modulo 8. We also examine the behavior of $ped_{-2}(n)$ modulo 8 in detail where $ped_{-2}(n)$ denotes the number of bipartitions of n with even parts distinct. As a result, we find infinite families of congruences for $ped_{-2}(n)$ modulo 8.

1. Introduction

Let $ped(n)$ denote the number of partitions of n wherein even parts are distinct (and odd parts are unrestricted). The generating function for $ped(n)$ [1] is

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \prod_{m=1}^{\infty} \frac{(1-q^{4m})}{(1-q^m)}. \tag{1.1}$$

Note that by (1.1), the number of partitions of n wherein even parts are distinct equals the number of partitions of n with no parts divisible by 4, i.e., the 4-regular partitions (see [1] and references therein). The arithmetic properties were studied by Andrews, Hirschhorn and Sellers [1] and Chen [4]. For example, in [1], Andrews, et al., proved that for all $n \geq 0$,

$$ped(9n+4) \equiv 0 \pmod 4 \tag{1.2}$$

and

$$ped(9n+7) \equiv 0 \pmod 4. \tag{1.3}$$

Suppose that r is an integer such that $1 \leq r < 8p$, $rp \equiv 1 (\text{mod } 8)$, and $(r,p) = 1$. By using modular forms, Chen [4] showed that if $c(p) \equiv 0 (\text{mod } 4)$, then, for all $n \geq 0$, $\alpha \geq 1$,

$$ped\left(p^{2\alpha}n + \frac{rp^{2\alpha-1}-1}{8}\right) \equiv 0 \pmod 4 \tag{1.4}$$

where $c(p)$ is the p-th coefficient of $\frac{\eta^4(16z)}{\eta(8z)\eta(32z)} := \sum_{n=1}^{\infty} c(n)q^n$. Note that in [4], Chen did not show the coefficients of $c(p)$ explicitly. In a beautiful paper [5], Chen studied arithmetic properties for the number of k-tuple partitions with even parts

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distinct modulo 2 for any positive integer k by using the nilpotence of Hecke operators mod 2. Berkovich and Patane [2] calculated $c(n)$ explicitly. In particular, they showed that $c(p) = 0$ if and only if $p = 2$, $p \equiv 5 \pmod 8$ and $p \equiv 3 \pmod 4$.

As a direct application of Chen's, Berkovich and Patane's theorems, we have the following.

Corollary 1.1. *Let p be a prime which is congruent to 5 modulo 8 or congruent to 3 modulo 4 and suppose that r is an integer such that $1 \le r < 8p$, $rp \equiv 1 \pmod 8$, and $(r, p) = 1$, and then*

$$ped\left(p^{2\alpha}n + \frac{rp^{2\alpha-1} - 1}{8}\right) \equiv 0 \pmod 4$$

for all $n \ge 0$ and $\alpha \ge 1$.

Ono and Penniston [9] showed an explicit formula for $Q(n)$ modulo 8 by using the arithmetic of the ring of $\mathbb{Z}[\sqrt{-6}]$ where $Q(n)$ denotes the number of partitions of an integer n into distinct parts. We are unable to explicitly determine $ped(n)$ modulo 8. But we can prove infinitely families congruences for $ped(n)$ modulo 8. Our first main result is the following.

Theorem 1.2. *Let p be a prime which is congruent to $7 \pmod 8$. Suppose that r is an integer such that $1 \le r < 8p$, $r \equiv 7 \pmod 8$, and $(r, p) = 1$, and then for all $n \ge 0$, $\alpha \ge 0$, we have*

$$ped\left(p^{2\alpha+2}n + \frac{rp^{2\alpha+1} + 1}{8}\right) \equiv 0 \pmod 8.$$

Example 1.3. For all $n \ge 0$, $\alpha \ge 0$,

$$ped\left(7^{2\alpha+2}n + \frac{r \times 7^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod 8,$$

for $r = 15, 23, 31, 39, 47$ and 55.

Let $ped_{-2}(n)$ be the number of bipartitions of n with even parts distinct. The generating function of $ped_{-2}(n)$ [7] is

$$\sum_{n=0}^{\infty} ped_{-2}(n)q^n = \frac{(q^4; q^4)_\infty^2}{(q; q)_\infty^2} = \prod_{m=1}^{\infty} \frac{(1 - q^{4m})^2}{(1 - q^m)^2}. \tag{1.5}$$

Recently in [7], Lin investigated arithmetic properties of $ped_{-2}(n)$. In particular, he showed following theorems.

Theorem 1.4 (*[7]*)**.** *For $\alpha \ge 0$ and any $n \ge 0$, we have*

$$ped_{-2}(n)\left(3^{2\alpha+2}n + \frac{11 \times 3^{2\alpha+1} - 1}{4}\right) \equiv 0 \pmod 3,$$

$$ped_{-2}(n)\left(3^{2\alpha+3}n + \frac{5 \times 3^{2\alpha+2} - 1}{4}\right) \equiv 0 \pmod 3.$$

Theorem 1.5 (*[7]*)**.** *$ped_{-2}(n)$ is even unless n is of the form $k(k + 1)$ for some $k \ge 0$. Furthermore, $ped_{-2}(n)$ is a multiple of 4 if n is not the sum of two triangular numbers.*

As a corollary of Theorems 1.4 and 1.5, Lin proved an infinite family of congruences for $ped_{-2}(n)$ modulo 12:

$$ped_{-2}\left(3^{2\alpha+2}n + \frac{11 \times 3^{2\alpha+1} - 1}{4}\right) \equiv 0 \pmod{12},$$

for any integers $\alpha \ge 0$ and $n \ge 0$.

As in [9], our second main achievement is to examine $ped_{-2}(n)$ modulo 8 in detail.

Theorem 1.6. *If n is a non-negative integer, then let N and M be the unique positive integers for which*

$$4n + 1 = N^2M$$

where M is square-free. Then the following are true.

(1) *If $M = 1$, then $ped_{-2}(n)$ is odd.*
(2) *If $M = p$, and $\mathrm{ord}_p(4n + 1) \equiv 1 \pmod 4$, then $ped_{-2}(n) \equiv 2 \pmod 4$.*
(3) *If $M = p$, and $\mathrm{ord}_p(4n + 1) \equiv 3 \pmod 8$, then $ped_{-2}(n) \equiv 4 \pmod 8$.*
(4) *If $M = p_1p_2$, where p_1 and p_2 are distinct primes, $p_i \equiv 1 \pmod 4$ and $\mathrm{ord}_{p_i}(4n + 1) \equiv 1 \pmod 4$ for $i = 1, 2$, then $ped_{-2}(n) \equiv 4 \pmod 8$.*
(5) *In all other cases we have that $ped_{-2}(n) \equiv 0 \pmod 8$.*

As a corollary to Theorem 1.6 we can show infinite families of congruences for $ped_{-2}(n)$ modulo 8.