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Adjacent vertex distinguishing indices of planar graphs without 3-cycles

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ABSTRACT

Given a proper edge k-coloring ϕ and a vertex $v \in V(G)$, let $C_\phi(v)$ denote the set of colors used on edges incident to v with respect to ϕ. The adjacent vertex distinguishing index of G, denoted by $\chi'_a(G)$, is the least value of k such that G has a proper edge k-coloring which satisfies $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v. In this paper, we show that if G is a connected planar graph with maximum degree $\Delta \geq 12$ and without 3-cycles, then $\Delta \leq \chi'_a(G) \leq \Delta+1$, and $\chi'_a(G) = \Delta+1$ if and only if G contains two adjacent vertices of maximum degree. This extends a result in Edwards et al. (2006), which says that if G is a connected bipartite planar graph with $\Delta \geq 12$ then $\chi'_a(G) \leq \Delta + 1$.

1. Introduction

Only simple and finite graphs are considered in this paper. A *plane* graph is a particular drawing of a planar graph on the Euclidean plane. Let $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ (for short Δ), $\delta(G)$, and $g(G)$ denote the vertex set, the edge set, the face set, the maximum degree, the minimum degree, and the girth of a given plane graph G, respectively. A *proper edge k-coloring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \dots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. We use $C_\phi(v)$ to denote the set of colors assigned to edges incident to a vertex v, i.e., $C_\phi(v) = \{\phi(uv) \mid uv \in E(G)\}$. The coloring ϕ is called an *adjacent vertex distinguishing edge coloring* or an *avd-coloring*, if $C_\phi(u) \neq C_\phi(v)$ for any pair of adjacent vertices u and v. Two adjacent vertices u and v are *conflict* if $C_\phi(u) = C_\phi(v)$. If there is no confusion in the context, we usually omit the letter ϕ in $C_\phi(v)$. A graph G is *normal* if it contains no isolated edges. Clearly, G has an avd-coloring if and only if G is normal. Thus, we always assume that G is normal in the following. The *adjacent vertex distinguishing chromatic index* $\chi'_a(G)$ of a graph G is the smallest integer k such that G has a k-avd-coloring.

Zhang, Liu and Wang [12] first introduced and investigated the adjacent vertex distinguishing edge coloring of graphs (named as *adjacent strong edge coloring*). Based on the argument for some special classes of graphs, they proposed the following challenging conjecture:

Conjecture 1. *If G is a connected graph with $|V(G)| \geq 3$ and $G \neq C_5$, then $\chi'_a(G) \leq \Delta + 2$.*

Balister et al. [1] confirmed Conjecture 1 for all bipartite graphs and graphs with $\Delta = 3$, and proved that if G is a normal graph with the (vertex) chromatic number k, then $\chi'_a(G) \leq \Delta + O(\log k)$. Using a powerful probabilistic method, Hatami [4] showed that every graph G with $\Delta > 10^{20}$ has $\chi'_a(G) \leq \Delta + 300$. Edwards et al. [3] proved that $\chi'_a(G) \leq \Delta + 1$ if G is

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a bipartite planar graph with $\Delta \geq 12$. Wang and Wang [8] verified Conjecture 1 for the class of graphs with the smaller maximum average degree, and their results were further extended by Hocquard and Montassier [5,6]. More recently, it was characterized in [9] that the adjacent vertex distinguishing chromatic index of a K_4-minor-free graph G with $\Delta \geq 5$ is equal to Δ or $\Delta + 1$. For planar graphs, Horňák et al. [7] proved that $\chi'_a(G) \leq \Delta + 2$ for planar graphs with maximum degree at least 12. Bu et al. [2] showed that if G is a planar graph with $g(G) \geq 6$, then $\chi'_a(G) \leq \Delta + 2$. This result was improved to the case $g(G) \geq 5$ [10]. Further, it was shown in [11] that if G is a planar graph with $g(G) \geq 4$ and $\Delta \geq 6$, then $\chi'_a(G) \leq \Delta + 2$.

In this paper, we will determine the adjacent vertex distinguishing chromatic indices of planar graphs with $\Delta \geq 12$ and without 3-cycles, which reinforces obviously the result of [3]. More precisely, we show the following.

Theorem 2. *Suppose that G is a connected planar graph without 3-cycles.*
(A) *Let $T(G) = \max\{12, \Delta + 1\}$, then $\chi'_a(G) \leq T(G)$.*
(B) *If $\Delta \geq 12$, then $\Delta \leq \chi'_a(G) \leq \Delta + 1$, and $\chi'_a(G) = \Delta + 1$ if and only if G contains two adjacent vertices of degree Δ.*

To obtain our main result, we need to introduce some notations. Suppose that H is a subgraph of a given plane graph G. For $x \in V(H) \cup F(H)$, let $d_H(x)$ denote the degree of x in H. For a face $f \in F(H)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \dots u_m]$ if $u_1, u_2, \dots, u_m$ are the vertices of $b(f)$ in clockwise order. Repeated occurrences of a vertex are allowed. A vertex of degree k (at least k, at most k, resp.) is called a *k-vertex* (*k^+-vertex*, *k^--vertex*, resp.). We use $N^H_k(v)$ to denote the set of k-vertices adjacent to v in H, and $n^H_k(v) = |N^H_k(v)|$. Similarly, we can define $n^H_{k^+}(v)$ and $n^H_{k^-}(v)$. Let $m^H_k(f)$ denote the number of k-vertices incident to a face f. If there is no confusion, we usually omit the letter G in $d_G(x)$, $n^G_k(x)$, $n^G_{k^+}(x)$ and $n^G_{k^-}(x)$. For two positive integers p, q with $p < q$, we use $[p, q]$ to denote the set of all integers between p and q (including p and q).

2. Proof of Theorem 2(A)

2.1. Unavoidable configurations

Our proof proceeds by *reductio ad absurdum*. Assume that G is a counterexample to Theorem 2(A) such that $|V(G)|+|E(G)|$ is as small as possible. Obviously, G is connected. We may assume that $\Delta \geq 4$ by a result in [1].

With a similar argument as for Claims 1–3 in [11], we can derive the following Claims 1–3.

Claim 1. *No 1-vertex is adjacent to a 6^--vertex.*

Remark 1. By Claim 1, for any edge $e \in E(G)$, $G - e$ is a normal planar graph without 3-cycles, and henceforth $\chi'_a(G - e) \leq T(G - e) \leq T(G)$ by the minimality of G.

Claim 2. *No k-vertex, with $2 \leq k \leq 3$, is adjacent to at least two k-vertices.*

A k-vertex v, $2 \leq k \leq 3$, is called *bad* if it is adjacent to exactly one k-vertex. A 4-cycle $x_1x_2x_3x_4x_1$ is called *bad* if $d(x_1) = d(x_3) = 2$.

Claim 3. *No 2-vertex is adjacent to a 3-vertex.*

In what follows, we say that a recoloring for $E(G)$ is *reasonable* if it gives a proper edge coloring and no pairs of adjacent conflicting vertices are produced. Let $C = \{1, 2, \dots, T(G)\}$ denote a set of $T(G)$ colors.

Claim 4. (1) *Every edge uv with $d(u) = d(v) = 2$ is in a 4-cycle.*
(2) *No 5-cycle contains three 2-vertices.*

Proof. (1) Assume to the contrary that uv is not in any 4-cycle. Let u_1 (resp., v_1) be the neighbor of u (resp., v) other than v (resp., u). Then $u_1 \neq v_1$ and $u_1v_1 \notin E(G)$ by the assumption. By Claims 1 and 2, $d(u_1), d(v_1) \geq 3$. Let $H = G - v + uv_1$. Then H is a simple normal planar graph without 3-cycles, which admits a $T(G)$-avd-coloring ϕ with the color set C by the minimality of G. Note that $\phi(uu_1) \neq \phi(uv_1)$. We extend ϕ to the whole graph G by coloring vv_1 with $\phi(uv_1)$ and uv with a color in $C \setminus \{\phi(uu_1), \phi(uv_1)\}$.

(2) Assume to the contrary that G contains a 5-cycle $v_1v_2 \cdots v_5v_1$ such that $d(v_1) = d(v_2) = d(v_4) = 2$ by Claim 2. By Claim 4(1), v_1v_2 is in some 4-cycle, i.e., $v_3v_5 \in E(G)$. It follows that G contains a 3-cycle $v_3v_4v_5v_3$, contradicting the assumption. □

Claim 5. *There is no edge $uv \in E(G)$ such that $d(u) \leq 3$ and $4 \leq d(v) \leq 5$.*

Proof. Assume to the contrary that G contains such an edge uv. Then $G - uv$ has a $T(G)$-avd-coloring ϕ using the color set C by Remark 1. Note that u has at most one conflict vertex by Claim 2 and v has at most four conflict vertices. Since $|C| \geq 12$ and $|C| - (d(u) - 1) - (d(v) - 1) \geq 6$, we may color uv with a color in $C \setminus (C(u) \cup C(v))$ such that neither u nor v conflicts with its neighbors. □

In the proof of Claims 6–9 and 11, we always assume that v is a k-vertex with neighbors $v_1, v_2, \dots, v_k$ such that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_k)$. If $d(v_i) = 2$, we use u_i to denote the neighbor of v_i different from v.