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Circle lattice point problem, revisited

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ABSTRACT

Let X be a compact region of area n in the plane. We show that if X is a strictly convex region, or a region bounded by an irreducible algebraic curve, then X can be translated to a position where it covers exactly n lattice points. If X is a polygon, or a convex region, then it can be rotated and translated so that it covers exactly n lattice points.

1. Introduction

In 1957, H. Steinhaus posed the following problem [6,8]: Is there a circle in the plane $\mathbb{R}^2$ that contains in its interior exactly n lattice points, for any given n? (A lattice point means a point whose coordinates are all integers.) W. Sierpinski [7] showed, by noting that the distances from the point $(\sqrt{2}, \frac{1}{3})$ to lattice points are all different, that such a circle can be obtained by adjusting the radius of a circle with center $(\sqrt{2}, \frac{1}{3})$. It seems that Steinhaus proved the following slightly stronger result, see Honsberger [1, p. 118].

Theorem 1 (*Steinhaus*). *If X is a circular disk of area n, then X can be translated so that it covers exactly n lattice points.*

It is impossible to replace "a circular disk" in this theorem by "a square". For example, consider the square $[0, \sqrt{3}] \times [0, \sqrt{3}]$ of area 3. When we translate this square, the number of lattice points covered by the square is clearly represented as $m \times n$, where $1 \le m, n \le 2$, that is, the number is either 1×1 or 1×2 or 2×2.

Now, by what kind of figures can we replace "a circular disk" in Theorem 1? We show the following.

(i) It is possible to replace "a circular disk" in Theorem 1 by "a strictly convex region" (i.e., a compact convex region whose boundary contains no line segment), and by "a region bounded by an irreducible algebraic curve".
(ii) For "a (non-strict) convex region" and "a polygon", similar results as Theorem 1 also hold if we allow rotations besides translations. Namely, if their areas are n, then they can be rotated and translated in $\mathbb{R}^2$ so that they cover exactly n lattice points.

The case of polygon is generalized to higher dimensions in [3]: Every d-dimensional polyhedron of volume n can be rotated and translated in $\mathbb{R}^d$ so that it contains exactly n lattice points.

Lattice points on a circle and quadratic curves in the plane are considered in [2,4,5].

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2. Main theorem

For a point set $X \subset \mathbb{R}^2$ and a point $v \in \mathbb{R}^2$, let $v + X$ denote the translate of X along $\vec{v}$, and X^* denote the set that is symmetric to X with respect to the origin O.

A planar curve C is called *lattice-generic* if $C \cap (p + C)$ is a finite set for every lattice point $p \neq O$. For example, circles and ellipses are lattice-generic. Note that if a curve C is lattice-generic, then $(p+C) \cap (q+C)$ is also a finite set for every distinct lattice points p, q. Indeed, $q - p \neq O$ implies that $C \cap (q - p + C)$ is a finite set, and hence its translate $(p + C) \cap (q + C)$ is also a finite set.

It is clear that if C is lattice-generic, then C^* is also lattice-generic. Note that for a curve C and two points $u, v \in \mathbb{R}^2$, we have

$$v \in u + C \Leftrightarrow v - u \in C \Leftrightarrow u - v \in C^* \Leftrightarrow u \in v + C^*. \quad (1)$$

The following is the main theorem in this paper.

Theorem 2. *If X is a compact region of area n bounded by a lattice-generic closed curve C, then it is possible to translate X to a position where it covers exactly n lattice points.*

To prove this theorem, we use Blichfeldt's lemma.

Lemma 1 (*Blichfeldt*)**.** *If a planar bounded region X has area n, then it is possible to translate X to a position where it covers at least n lattice points, and it is also possible to translate X to a position where it covers at most n lattice points.*

Intuitive proofs of this lemma are given in Honsberger [1] and Steinhaus [9].

Proof of Theorem 2. Let X° denote the interior of X. Since $area(X^\circ) = area(X) = n$, it follows from Lemma 1 that there are $u_0, u_1 \in \mathbb{R}^2$ such that

$$|(u_0 + X) \cap \mathbb{Z}^2| \leq n, \qquad |(u_1 + X^\circ) \cap \mathbb{Z}^2| \geq n. \quad (2)$$

Let Q be a square that contains u_0, u_1. The set S defined by

$$S = \{p \in \mathbb{Z}^2 : (p + C^*) \cap Q \neq \emptyset\}$$

is a finite set. Since C^* is also lattice-generic, the set F defined by

$$F = \bigcup \{Q \cap (p + C^*) \cap (q + C^*) : p, q \in S, p \neq q\}$$

is also a finite set. Notice that since X is compact, the minimum distance δ from a lattice point lying in the exterior of X to C is positive. Hence for any point u within the distance $\delta/2$ from u_0, we have $|(u + X) \cap \mathbb{Z}^2| \leq n$. So, by replacing u_0 with an appropriate point near to u_0 if necessary, we may suppose $u_0 \notin F$. Similarly, we may suppose that $u_1 \notin F$. Since $Q - F$ is path-connected, we can connect u_0 and u_1 by a simple curve J in $Q - F$. Note that if $p, q \in x + C$ for some $p, q \in \mathbb{Z}^2, p \neq q$, then $x \in (p + C^*) \cap (q + C^*)$ by (1). Hence, if $x \in J$, then at most one lattice point lies on $x + C$. Therefore, when x moves along J from u_0 to u_1, the number of lattice points $|(x+X) \cap \mathbb{Z}^2|$ changes one by one. Therefore, we can deduce from (2) that there exists $x \in J$ such that $|(x + X) \cap \mathbb{Z}^2| = n$. □

3. Application

3.1. Plane algebraic curves

An (affine plane) algebraic curve in $\mathbb{R}^2$ is the set defined by an equation $f(x, y) = 0$, where $f(x, y) \in \mathbb{R}[x, y]$. For example, quadratic curves are algebraic curves. If C is an algebraic curve, then so are $v + C$ and C^*. If $f(x, y)$ is irreducible in $\mathbb{R}[x, y]$, then the algebraic curve defined by f is called an *irreducible algebraic curve*. If C is an irreducible algebraic curve, then so are $v + C$ and C^*. By Bézout's theorem, if two irreducible algebraic curves share infinitely many points in common, then the curves coincide completely. From this it follows that irreducible algebraic curves other than lines are lattice-generic.

Corollary 1. *If a compact region bounded by an irreducible algebraic curve has area n, then it is possible to translate X to a position where it covers exactly n lattice points.*

3.2. Polygons

Note that a polygon is not necessarily a convex polygon. Let $SO(2)$ denote the rotation group of the plane $\mathbb{R}^2$ around the origin. Each $\sigma \in SO(2)$ is a linear transformation of $\mathbb{R}^2$.