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Bounds for generalized Sidon sets

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ABSTRACT

Let Γ be an abelian group and $g \geq h \geq 2$ be integers. A set $A \subset \Gamma$ is a $C_h[g]$-set if given any set $X \subset \Gamma$ with $|X| = h$, and any set $\{k_1, \ldots, k_g\} \subset \Gamma$, at least one of the translates $X + k_i$ is not contained in A. For any $g \geq h \geq 2$, we prove that if $A \subset \{1, 2, \ldots, n\}$ is a $C_h[g]$-set in $\mathbb{Z}$, then $|A| \leq (g-1)^{1/h} n^{1-1/h} + O(n^{1/2-1/2h})$. We show that for any integer $n \geq 1$, there is a $C_3[3]$-set $A \subset \{1, 2, \ldots, n\}$ with $|A| \geq (4^{-2/3} + o(1)) n^{2/3}$. We also show that for any odd prime p, there is a $C_3[3]$-set $A \subset \mathbb{F}_p^3$ with $|A| \geq p^2 - p$, which is asymptotically best possible. Using the projective norm graphs from extremal graph theory, we show that for each integer $h \geq 3$, there is a $C_h[h!+1]$-set $A \subset \{1, 2, \ldots, n\}$ with $|A| \geq (c_h + o(1)) n^{1-1/h}$. A set A is a weak $C_h[g]$-set if we add the condition that the translates $X + k_1, \ldots, X + k_g$ are all pairwise disjoint. We use the probabilistic method to construct weak $C_h[g]$-sets in $\{1, 2, \ldots, n\}$ for any $g \geq h \geq 2$. Lastly we obtain upper bounds on infinite $C_h[g]$-sequences. We prove that for any infinite $C_h[g]$-sequence $A \subset \mathbb{N}$, we have $A(n) = O(n^{1-1/h} (\log n)^{-1/h})$ for infinitely many n, where $A(n) = |A \cap \{1, 2, \ldots, n\}|$.

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1. Introduction

Given an integer $n \geq 1$, write $[n]$ for $\{1, 2, \ldots, n\}$. Let Γ be an abelian group and $g \geq h \geq 2$ be integers. A set $A \subset \Gamma$ is a $C_h[g]$-*set* if given any set $X \subset \Gamma$ with $|X| = h$, and any set $\{k_1, \ldots, k_g\} \subset \Gamma$, at least one of the translates

$$X + k_i := \{x + k_i : x \in X\}$$

is not contained in A. These sets were introduced by Erdős and Harzheim in [8], and they are a natural generalization of the well-studied Sidon sets. A Sidon set is a $C_2[2]$-set. We will always assume that $g \geq h \geq 2$. The reason for this is that if $X = \{x_1, \ldots, x_h\}$ and $K = \{k_1, \ldots, k_g\}$, then A contains each of the translates $X + k_1, \ldots, X + k_g$ if and only if A contains each of the translates $K + x_1, \ldots, K + x_h$.

Our starting point is a connection between $C_h[g]$-sets and the famous Zarankiewicz problem from extremal combinatorics. Given integers m, n, s, t with $m \geq s \geq 1$ and $n \geq t \geq 1$, let $z(m, n, s, t)$ be the largest integer N such that there is an $m \times n$ 0–1 matrix M, that contains N 1's, and does not contain an $s \times t$ submatrix of all 1's. Determining $z(m, n, s, t)$ is known as the problem of Zarankiewicz.

Proposition 1. *Let Γ be a finite abelian group of order n. Let $A \subset \Gamma$ and let $g \geq h \geq 2$ be integers. If A is a $C_h[g]$-set in Γ, then*

$$n|A| \leq z(n, n, g, h). \tag{1}$$

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To see this, let $A \subset \Gamma$ be a $C_h[g]$-set where $\Gamma = \{b_1, \ldots, b_n\}$ is a finite abelian group of order n. Define an $n \times n$ 0–1 matrix M by putting a 1 in the (i, j)-entry if $b_i + b_j \in A$, and 0 otherwise. A $g \times h$ submatrix of all 1's consists of a set $X = \{x_1, \ldots, x_h\}$ of h distinct elements of Γ, and a sequence $k_1, \ldots, k_g$ of g distinct elements of Γ, such that $x_i + k_j \in A$ for all $1 \le i \le h$, $1 \le j \le g$. There is no such submatrix since A is a $C_h[g]$-set. Furthermore, each row of M contains $|A|$ 1's so that $n|A| \le z(n, n, g, h)$.

Füredi [10] proved that

$$z(m, n, s, t) \le (s - t + 1)^{1/2} n m^{1-1/t} + t m^{2-2/t} + tn \tag{2}$$

for any integers $m \ge s \ge t \ge 1$ and $n \ge t$. Therefore, if $A \subset \Gamma$ is a $C_h[g]$-set and Γ is a finite abelian group of order n, then

$$|A| \le (g - h + 1)^{1/h} n^{1-1/h} + h n^{1-2/h} + h. \tag{3}$$

If $A \subset [n]$ is a $C_h[g]$-set, then it is not difficult to show that A is a $C_h[g]$-set in $\mathbb{Z}_{2n}$, thus by (3),

$$|A| \le (g - h + 1)^{1/h} 2^{1-1/h} n^{1-1/h} + h(2n)^{1-2/h} + h.$$

Our first result improves this upper bound.

Theorem 1. *If $A \subset [n]$ is a $C_h[g]$-set with $g \ge h \ge 2$, then*

$$|A| \le (g - 1)^{1/h} n^{1-1/h} + O\left(n^{1/2-1/2h}\right). \tag{4}$$

This theorem is a refinement of the estimate $|A| = O(n^{1-1/h})$ proved by Erdős and Harzheim [8]. Recall that $C_2[2]$-sets are Sidon sets. Theorem 1 recovers the well-known upper bound for the size of Sidon sets in $[n]$ obtained by Erdős and Turán [9]. In general, $C_2[g]$-sets are those sets A such that each nonzero difference $a - a'$ with $a, a' \in A$ appears at most $g - 1$ times. Theorem 1 recovers Corollary 2.1 in [4].

If $A \subset [n]$ is a Sidon set, then for any $g \ge 2$, A is a $C_2[g]$-set. There are Sidon sets $A \subset [n]$ with $|A| = (1 + o(1))n^{1/2}$ thus the exponent of (4) is correct when $h = 2$. Motivated by constructions in extremal graph theory, we can show that the exponent of (4) is correct for other values of h.

Theorem 2. *Let p be an odd prime and $\alpha \in \mathbb{F}_p$ be chosen to be a quadratic non-residue if $p \equiv 1 \pmod 4$, and a nonzero quadratic residue otherwise. The set*

$$A = \{(x_1, x_2, x_3) \in \mathbb{F}_p^3 : x_1^2 + x_2^2 + x_3^2 = \alpha\}$$

is a $C_3[3]$-set in the group $\mathbb{F}_p^3$ with $|A| \ge p^2 - p$.

Corollary 1. *For any integer $n \ge 1$, there is a $C_3[3]$-set $A \subset [n]$ with*

$$|A| \ge \left(4^{-2/3} + o(1)\right) n^{2/3}.$$

By (3), Theorem 2 is asymptotically best possible. It is an open problem to determine the maximum size of a $C_3[3]$-set in $[n]$.

Proposition 1 suggests that the methods used to construct $K_{g,h}$-free graphs may be used to construct $C_h[g]$-sets. Using the norm graphs of Kollár, Rónyai, and Szabó [11], we construct $C_h[h! + 1]$-sets $A \subset [n]$ with $|A| \ge c_h n^{1-1/h}$ for each $h \ge 2$.

Theorem 3. *Let $h \ge 2$ be an integer. For any integer n, there is a $C_h[h! + 1]$-set $A \subset [n]$ with*

$$|A| = (1 + o(1)) \left(\frac{n}{2^{h-1}}\right)^{1-1/h}.$$

Using the probabilistic method we can construct sets that are almost $C_h[g]$ for all $g \ge h \ge 2$. A set $A \subset \Gamma$ is a *weak $C_h[g]$-set* if given any set $X \subset \Gamma$ with $|X| = h$, and any set $\{k_1, \ldots, k_g\} \subset \Gamma$ such that $X + k_1, \ldots, X + k_g$ are all pairwise disjoint, at least one of the translates $X + k_i$ is not contained in A. Erdős and Harzheim used the probabilistic method to construct weak $C_h[g]$-sets with $|A| \gg n^{(1-\frac{1}{h})(1-\frac{1}{g})}$. Here we use the alteration method to improve the exponent.

Theorem 4. *For any integers $g \ge h \ge 2$, there exists a weak-$C_h[g]$-set $A \subset [n]$ such that*

$$|A| \ge \frac{1}{8} n^{\left(1-\frac{1}{h}\right)\left(1-\frac{1}{g}\right)\left(1+\frac{1}{hg-1}\right)}.$$

It should be noted that for h fixed, Theorem 4 gives $|A| \ge n^{1-\frac{1}{h}-\epsilon}$ for g sufficiently large, being a lower bound close to the exponent given in Theorem 1.

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