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Geometric constructions of two-character sets

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ABSTRACT

A two-character set in a finite projective space is a set of points with the property that the intersection number with any hyperplanes only takes two values. In this paper constructions of some two-character sets are given. In particular, infinite families of tight sets of the symplectic generalized quadrangle $\mathcal{W}(3, q^2)$ and the Hermitian surface $\mathcal{H}(3, q^2)$ are provided. A quasi-Hermitian variety $\mathcal{H}$ in $PG(r, q^2)$ is a combinatorial generalization of the (non-degenerate) Hermitian variety $\mathcal{H}(r, q^2)$ so that $\mathcal{H}$ and $\mathcal{H}(r, q^2)$ have the same number of points and the same intersection numbers with hyperplanes. Here we construct two families of quasi-Hermitian varieties, for r, q both odd, admitting $\mathrm{P\Gamma O}^+(r+1, q)$ and $\mathrm{P\Gamma O}^-(r+1, q)$ as automorphisms group.

1. Introduction

A *strongly regular graph* $srg(v, k, \lambda, \mu)$ is a regular graph such that there are constants λ and μ with the property that every pair of adjacent vertices has λ common neighbours and every pair of non-adjacent vertices has μ common neighbours. A *two-character set* in the projective space $PG(r, q)$ is a set $\mathcal{S}$ of n points with the property that every hyperplane meets $\mathcal{S}$ in either $n - w_1$ or $n - w_2$ points and generates $PG(r, q)$. Then the positive integers w_1 and w_2 are called the *weights* of the two-character set. Embed $PG(r, q)$ as a hyperplane H in $PG(r+1, q)$. The *linear representation graph* $\Gamma_r^*(\mathcal{S})$ is the graph having as vertices the points of $PG(r+1, q) \setminus H$ and in which two vertices are adjacent whenever the line defined by them meets $\mathcal{S}$. It follows that $\Gamma_r^*(\mathcal{S})$ has $v = q^{r+1}$ vertices and valency $k = (q-1)n$. Delsarte [8] proved that this graph is strongly regular if $\mathcal{S}$ is a two-character set. The other parameters of the graph $\Gamma_r^*(\mathcal{S})$ are $\lambda = k - 1 + (k - qw_1 + 1)(k - qw_2 + 1)$ and $\mu = k + (k - qw_1)(k - qw_2)$. On the other hand, by regarding the coordinates of the elements of $\mathcal{S}$ as columns of the generator matrix of a code $\mathcal{L}$ of length n and dimension $r + 1$, the two-character set property of $\mathcal{S}$ translates into the fact that the code $\mathcal{L}$ has two weights (w_1 and w_2) [5]. Such a code is said to be a *projective two-weight code*. The weights of the code are precisely the weights of the two-character set.

A *generalized quadrangle of order* (s, t) ($GQ(s, t)$ for short) is an incidence structure $\mathcal{Q} = (P, B, I)$ of points and lines with the properties that any two points (lines) are incident with at most one line (point), every point is incident with $t + 1$ lines, every line is incident with $s + 1$ points, and for any point P and line l which are not incident, there is a unique point on l collinear with P. The standard reference is [13].

Here we will focus on the (classical) generalized quadrangles $\mathcal{W}(3, q)$ and $\mathcal{H}(3, q^2)$. The first of these is the incidence structure of all totally isotropic points and totally isotropic lines with respect to a (non-degenerate) symplectic polarity of $PG(3, q)$. It is a generalized quadrangle of order (q, q) with automorphism group $\mathrm{P\Gamma Sp}(4, q)$. The generalized quadrangle $\mathcal{H}(3, q^2)$ of order (q^2, q) is the incidence structure of all points and lines of a non-singular Hermitian variety of $PG(3, q^2)$. Its automorphism group is $\mathrm{P\Gamma U}(4, q^2)$, see [15] for more details. A set $\mathcal{T}$ of points of a generalized quadrangle of order (s, t)

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is an *i-tight set*, if for every point $P \in \mathcal{T}$, there are $s+i$ points of $\mathcal{T}$ collinear with P, and for every point $P \notin \mathcal{T}$, there are i points of $\mathcal{T}$ collinear with P. A set of points is *tight* if it is i-tight for some i. An *m-ovoid* of a generalized quadrangle $\mathcal{Q}$ is a set of points of $\mathcal{Q}$ meeting every line of $\mathcal{Q}$ in exactly m points. The dual concept of an m-ovoid is an m-cover, that is, a set of lines of $\mathcal{Q}$, say $\mathcal{L}$, such that every point of $\mathcal{Q}$ lies on exactly m lines of $\mathcal{L}$. Tight sets and m-ovoids of generalized quadrangles (polar spaces) are well studied objects, as well as their connections with two-character sets [12,4,3]. In Section 2, we prove that an m-cover of $\mathcal{W}(3,q)$ gives rise to tight sets of $\mathcal{W}(3,q^2)$ and $\mathcal{H}(3,q^2)$. An i-tight set of $\mathcal{W}(3,q^2)$ (or $\mathcal{H}(3,q^2)$) is a two-character set, where the intersection numbers with hyperplanes are i and q^2+i.

In the projective space $PG(r,q^2)$ over the finite field $GF(q^2)$, a *quasi-Hermitian variety* is a two-character set which has the same size and the same intersection numbers with hyperplanes as the non-degenerate Hermitian variety $\mathcal{H}(r,q^2)$. Obviously, a Hermitian variety can be viewed as a classical quasi-Hermitian variety. In [9], the authors constructed two infinite families of non-classical quasi-Hermitian varieties. Both consist of quasi-Hermitian varieties arising from $\mathcal{H}(r,q^2)$ by modifying some point–hyperplane incidences at the points in just one tangent space to the Hermitian variety $\mathcal{H}(r,q^2)$. Other instances of non-classical quasi-Hermitian varieties can be found in [1,2]. The essential idea in [1,2] is to modify many point–hyperplane incidences of the Hermitian variety $\mathcal{H}(r,q^2)$ by using a suitable quadratic transformation φ. In particular, φ maps $\mathcal{H}(r,q^2)$ to a quasi-Hermitian variety of a nonstandard model of $PG(r,q^2)$. As far as we know, these are the only known quasi-Hermitian varieties.

In Section 3, we construct two infinite families of non-classical quasi-Hermitian varieties in $PG(2n+1,q^2)$, q odd. Our quasi-Hermitian varieties have a completely different geometry from those constructed in [9,2,1]. In particular, they are associated in a natural way with the non-singular quadrics (of hyperbolic type or of elliptic type) lying in a Baer subgeometry Σ of $PG(2n+1,q^2)$ isomorphic to $PG(2n+1,q)$. Indeed our quasi-Hermitian varieties are obtained by taking all points on extended lines of Σ which are either tangent to or contained in a non-singular Baer quadric of Σ.

Quasi-Hermitian varieties in $PG(2n+1,q^2)$ are two-character sets, where the characters, that is the intersection numbers with hyperplanes, are

$$\frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1},$$

and

$$\frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1}+q^{2n}.$$

Our notation and terminology are standard. For generalities on quadrics and Hermitian varieties in projective spaces, the reader is referred to [11,15].

2. Tight sets

We summarize some properties of tight sets of a generalized quadrangle $\mathcal{S}$ of order (s,t), which have been proved in [12].

Let $\mathcal{A}$ and $\mathcal{B}$ be i-tight and j-tight sets of points of $\mathcal{S}$, respectively. Then:

(i) the size of an i-tight set of $\mathcal{S}$ is $i(s+1)$,
(ii) if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{B} \setminus \mathcal{A}$ is $(j-i)$-tight,
(iii) if $\mathcal{A}$ and $\mathcal{B}$ are disjoint, then $\mathcal{A} \cup \mathcal{B}$ is $(i+j)$-tight,
(iv) the set of points of $\mathcal{S}$ forms an $(st+1)$-tight set, the empty set is 0-tight and the set of points on a line is 1-tight.

Let $\mathcal{H}(3,q^2)$ be a (non-degenerate) Hermitian surface of $PG(3,q^2)$. It is well known that there exists a group G isomorphic to $\mathrm{PSp}(4,q)$ embedded in $\mathrm{P\Gamma U}(4,q^2)$ as a subfield subgroup, stabilizing a symplectic subquadrangle $\mathcal{W}(3,q) \subseteq \Sigma$, embedded in $\mathcal{H}(3,q^2)$. Here Σ is a Baer subgeometry of $PG(3,q^2)$ isomorphic to $PG(3,q)$.

Furthermore, $\mathcal{W}(3,q)$ is also embedded as a subquadrangle in a (unique) symplectic quadrangle of $PG(3,q^2)$, say $\mathcal{W}(3,q^2)$. In particular, the points of $PG(3,q^2)$ lying on the extended lines of $\mathcal{W}(3,q)$ are exactly the points of the Hermitian surface $\mathcal{H}(3,q^2)$, see [15]. From [3, Theorem 8] the group G has three orbits on points of $PG(3,q^2)$, that are $\mathcal{W}(3,q)$, $\mathcal{H}(3,q^2) \setminus \mathcal{W}(3,q)$ and $\mathcal{W}(3,q^2) \setminus \mathcal{H}(3,q^2)$ and each orbit is a tight set of $\mathcal{W}(3,q^2)$.

Instances of tight sets of the Hermitian surface $\mathcal{H}(3,q^2)$ are the points covered by partial spreads of $\mathcal{H}(3,q^2)$, embedded symplectic spaces $\mathcal{W}(3,q)$, and those obtained by taking complements and unions of these. In what follows we show that there exist tight sets of $\mathcal{H}(3,q^2)$ which cannot be constructed this way.

Let $\mathcal{L}$ be an m-cover of $\mathcal{W}(3,q)$, i.e., a set of $m(q^2+1)$ lines of $\mathcal{W}(3,q)$ such that every point of $\mathcal{W}(3,q)$ lies on exactly m lines of $\mathcal{L}$. For every line ℓ of $PG(3,q)$ we denote by $\bar{\ell}$ its extension in $PG(3,q^2)$. Let $\bar{\mathcal{L}}$ be the set of points of $PG(3,q^2)$ lying on the extended lines of $\mathcal{L}$. Notice that Σ is contained in $\bar{\mathcal{L}}$.

Theorem 2.1. *Let $\mathcal{L}$ be an m-cover of $\mathcal{W}(3,q)$, then*

- *$\bar{\mathcal{L}}$ is an $(m(q^2-q)+q+1)$-tight set of $\mathcal{W}(3,q^2)$,*
- *$\bar{\mathcal{L}}$ is an $(m(q^2-q)+q+1)$-tight set of $\mathcal{H}(3,q^2)$.*