Discrete Mathematics 333 (2014) 39–55

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Ordered multiplicity lists for eigenvalues of symmetric matrices whose graph is a linear tree[☆]

Charles R. Johnson [a], Andrew A. Li [b,*], Andrew J. Walker [c]

[a] Department of Mathematics, College of William and Mary, Williamsburg, VA 23185, USA

[b] Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027, USA

[c] Department of Mathematics, Duke University, Durham, NC 27708, USA

ARTICLE INFO

Article history:
Received 6 August 2012
Received in revised form 28 April 2014
Accepted 29 April 2014
Available online 28 June 2014

Keywords:
Linear tree
Symmetric matrix
Eigenvalue
Multiplicity

ABSTRACT

We consider the class of trees for which all vertices of degree at least 3 lie on a single induced path of the tree. For such trees, a new superposition principle is proposed to generate all possible ordered multiplicity lists for the eigenvalues of symmetric (Hermitian) matrices whose graph is such a tree. It is shown that no multiplicity lists other than these can occur and that for two subclasses all such lists do occur. Important contrasts with trees outside the class are given, and it is shown that several prior conjectures about multiplicity lists, including the Degree Conjecture, follow from our superposition principle.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

Given an undirected graph on n vertices G, we say that an n-by-n real symmetric matrix $A = (a_{ij})$ has graph G if for all $i \neq j$, there is an edge between vertices i and j if and only if $a_{ij} \neq 0$. We are interested in the spectra of all n-by-n real symmetric matrices whose graph is a given G, and in particular, in the eigenvalue multiplicities that occur. When a real symmetric matrix A has been identified for some graph G, we will commonly refer to the eigenvalues of G or some subgraph of G, by which we mean the eigenvalues of A or of the principle submatrix of A whose rows and columns correspond to the vertices of the subgraph. For any real symmetric matrix, we refer to the *ordered multiplicities* as the list obtained by arranging the distinct eigenvalues in increasing order and listing their multiplicities. We may also arrange the multiplicities in non-increasing order, a list that we call the *unordered multiplicities*. The set of ordered (unordered) multiplicity lists of real symmetric matrices whose graph is G is denoted $\mathcal{L}_o(G)$ ($\mathcal{L}_u(G)$).

Our focus here will be upon trees, or connected graphs with $n-1$ edges, and upon ordered multiplicity lists. (Since every complex Hermitian matrix whose graph is a tree is diagonally unitarily similar to a real symmetric matrix with the same graph, there is no loss of generality in considering only symmetric matrices in place of Hermitian matrices.) Multiplicity lists for certain classes of trees [8,9,13] and for trees on fewer than 12 vertices have been determined previously. Referring to any vertex of degree 3 or more as *high degree*, we consider a very rich class of trees, the linear trees (all high degree vertices lie on a path—see Definition 8), that includes some previously studied infinite classes of trees, as well as all but a few of the trees on fewer than 12 vertices. For linear trees, a combinatorial technique (involving a superposition principle) to generate all ordered multiplicity lists is proposed. The necessity of this proposal is proven in general and sufficiency is proven in

☆ This work was partially supported by NSF Grant DMS-0751964.

* Corresponding author.
E-mail addresses: crjohnso@math.wm.edu (C.R. Johnson), aali@mit.edu (A.A. Li), ajw32@duke.edu (A.J. Walker).

http://dx.doi.org/10.1016/j.disc.2014.04.030
0012-365X/© 2014 Elsevier B.V. All rights reserved.

two major cases: (a) when there are just two high degree vertices and (b) when the linear tree is depth 1. The latter uses a new application of the implicit function theorem that leads to an independently interesting class of matrices as Jacobians. The superposition principle generalizes that of [9] used to treat double generalized stars. There seems to be no simpler way to describe all such multiplicity lists. Finally, the results given are used to verify special cases of some outstanding conjectures.

2. Background

For convenience, we use the standard submatrix notation. Given an index set $\alpha \subseteq \{1, \ldots, n\}$, we denote the principle submatrix of A lying in rows and columns $\{1, \ldots, n\} - \alpha$ (respectively, α) by $A(\alpha)$ (respectively, $A[\alpha]$). Additionally, $A(\{i\})$ is abbreviated by $A(i)$. If A is a matrix of graph G, we may use a subgraph of G to specify an index set. For example, $A[G]$ is simply the matrix A. For any real number λ, we use $m_A(\lambda)$ to denote the multiplicity of λ as an eigenvalue of the matrix A.

The classical Interlacing Theorem is very important to our discussion. We state it here and refer the reader to [4] for a more thorough description.

Theorem 1 (*Interlacing Theorem*)**.** *Let A be an n-by-n Hermitian matrix with (real) eigenvalues*

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

and suppose $A(i)$ has eigenvalues

$$\mu_1 \le \mu_2 \le \cdots \le \mu_{n-1}$$

for some $i \in \{1, \ldots, n\}$. Then the eigenvalues satisfy the following inequalities:

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \cdots \le \mu_{n-1} \le \lambda_n.$$

A result following immediately from these inequalities is that, for any λ and $i = 1, \ldots, n$,

$$m_A(\lambda) - 1 \le m_{A(i)}(\lambda) \le m_A(\lambda) + 1.$$

In the case of trees, we have a very useful theorem coming from previous work in [14,15]. We state it without proof in the general form developed in [10].

Theorem 2. *Let A be a Hermitian matrix whose graph is a tree T, and suppose that there exists a vertex v of T and a real number λ such that λ is an eigenvalue of both A and $A(v)$. Then*

1. *there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$;*
2. *if $m_A(\lambda) \ge 2$, then v' may be chosen so that deg $v' \ge 3$ and so that there are at least three components T_1, T_2, and T_3 of $T - v'$ such that $m_{A[T_i]}(\lambda) \ge 1$, $i = 1, 2, 3$;*
3. *if $m_A(\lambda) = 1$, then v' may be chosen so that deg $v' \ge 2$ and so that there are two components T_1 and T_2 of $T - v'$ such that $m_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

For any tree T, we define the *path cover number* of T to be the minimum number of induced paths of T that cover all vertices without intersecting. We also define the *diameter* to be the maximum length of an induced path of T, where the length of a path refers to the number of nodes in the path. Note that this definition of length differs from the standard definition which defines the length of a path as the number of edges (one less than the number of nodes), but throughout we will continue to use the modified definition of length to remain consistent with previous references (e.g. [7,9]).

To demonstrate the relationship between a graph's structure and its multiplicity lists, we present the following two theorems from [6,7], respectively:

Theorem 3. *Given a tree T, the maximum multiplicity for any single eigenvalue in the multiplicity lists for T is equal to the path cover number of T.*

Theorem 4. *Given a tree T, the minimum number of distinct eigenvalues among the real symmetric matrices with graph T is at least the maximum number of nodes in an induced path of T.*

Let $l = (l_1, \ldots, l_a)$ be a partition of some integer N, i.e., each l_i is a positive integer and $l_1 + \cdots + l_a = N$. Two concepts about partitions will be needed. First, we denote by $l^* = (l_1^*, \ldots, l_b^*)$ the *conjugate partition* of l, so l_j^* is the number of i's such that $l_i \ge j$. Note that l^* is a partition of N with $l_1^* \ge \cdots \ge l_j^*$. The second concept is *majorization*. Let $u = (u_1, \ldots, u_c)$, $u_1 \ge \cdots \ge u_c$, and $v = (v_1, \ldots, v_d)$, $v_1 \ge \cdots \ge v_d$, be ordered partitions of M and N, respectively. Suppose $u_1 + \cdots + u_s \le v_1 + \cdots + v_s$ for all s, where u_s or v_s are interpreted as 0 when s is greater than c or d, respectively. Then we say that v *majorizes* u if $M = N$, denoting it as $u \preceq v$. If $M < N$, we create a partition u_e of N by appending $N - M$ 1's to u, and then we have $u_e \preceq v$.

We define a *generalized star* as a tree with at most one vertex of high degree. For each generalized star, we call the high degree vertex the *central vertex*, or let any vertex be the central vertex if there are none of high degree. The possible spectra of matrices whose graph is a generalized star were fully characterized in [9]. An additional result for generalized stars that we will use concerns the possible *upward* multiplicity lists. For a real symmetric matrix A with graph G, fix a vertex v. Then λ is an *upward eigenvalue* of A at v if $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, and the multiplicity of λ is called an *upward multiplicity*of A

Download English Version:

https://daneshyari.com/en/article/4647103

Download Persian Version:

https://daneshyari.com/article/4647103

Daneshyari.com