Discrete Mathematics 333 (2014) 66–83

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Degree sum conditions for the circumference of 4-connected graphs

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ARTICLE INFO

Article history:
Received 19 August 2013
Received in revised form 9 June 2014
Accepted 11 June 2014
Available online 30 June 2014

Keywords:
Cycle
Circumference
Degree sum
Connectivity
Independence number

ABSTRACT

We denote the order, the independence number, the connectivity and the minimum degree sum of independent four vertices of a graph G by $n(G)$, $\alpha(G)$, $\kappa(G)$ and $\sigma_4(G)$, respectively. The circumference of a graph G, denoted by $c(G)$, is the length of a longest cycle in G. We call a cycle C of a graph G a D_k-cycle if the order of each component of $G-V(C)$ is at most $k-1$. Our goal is to accomplish the proof of the statement that if G is a 4-connected graph, then $c(G) \geq \min\{\sigma_4(G)-\kappa(G)-\alpha(G)+1, n(G)\}$. In order to prove this, we consider three conditions for the construction of the outside of a longest cycle: (i) If G is a 3-connected graph and every longest cycle of G is a D_2-cycle, then $c(G) \geq \min\{\sigma_4(G)-\kappa(G)-\alpha(G)+1, n(G)\}$. (ii) If G is a 3-connected graph and every longest cycle is a D_3-cycle and some longest cycle is not a D_2-cycle, then $c(G) \geq \sigma_4(G)-\kappa(G)-4$. (iii) If G is a 4-connected graph and some longest cycle is not a D_3-cycle, then $c(G) \geq \sigma_4(G)-8$. For each condition, the lower bound of the circumference is sharp.

1. Introduction

All graphs considered in this paper are finite undirected simple graphs with at least three vertices. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [4].

A *Hamiltonian cycle* of a graph is a cycle containing all the vertices of the graph. A graph having a Hamiltonian cycle is called *Hamiltonian*. We can consider several generalizations of the hamiltonicity, for example, the cyclability and the circumference. A set $S \subseteq V(G)$ of vertices is *cyclable* in a graph G if G contains a cycle passing through all the vertices of S. The length of a longest cycle in a graph G is called *the circumference* of G.

In [15], Ozeki and the third author gave degree sum conditions for the hamiltonicity, and the cyclability. The purpose of this paper is to investigate whether there is a similar degree sum condition for the circumference.

Before that, we give the definition and the notation. Let G be a graph and let $S \subseteq V(G)$. The *order* of G is the number of vertices of G, denoted by $n(G)$. We denote by $G[S]$ the subgraph induced by S. We define $\alpha(S)$ as the maximum cardinality of an independent set of $G[S]$. For $x, y \in V(G)$, the *local connectivity* $\kappa(x, y)$ is defined to be the maximum number of internally-disjoint paths connecting x and y in G. We define $\kappa(S) = \min\{\kappa(x, y) : x, y \in S, x \neq y\}$. If $\alpha(S) \geq k$, let

$$\sigma_k(S) = \min\left\{\sum_{x\in X} d_G(x) \colon X \text{ is an independent set of } G[S] \text{ with } |X| = k\right\};$$

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http://dx.doi.org/10.1016/j.disc.2014.06.012
0012-365X/

otherwise $\sigma_k(S) = +\infty$. We simply write n instead of $n(G)$. If $S = V(G)$, we simply write α, κ and σ_k instead of $\alpha(V(G))$, $\kappa(V(G))$ and $\sigma_k(V(G))$, respectively. We call α and κ the *independence number* and the *connectivity*, respectively.

In 1960, Ore introduced a degree sum condition for a graph to be Hamiltonian.

Theorem A (*Ore [14]*). *Let G be a graph. If $\sigma_2 \geq n$, then G is Hamiltonian.*

In 1972, Chvátal and Erdős showed the relationship between the connectivity, the independence number and the hamiltonicity.

Theorem B (*Chvátal and Erdős [7]*). *Let G be a graph. If $\alpha \leq \kappa$, then G is Hamiltonian.*

From this theorem, we should consider degree sum conditions ensuring the existence of Hamiltonian cycles for graphs satisfying $\alpha \geq \kappa + 1$. Fraisse and Jung [8] gave a σ_2 condition for such graphs, which is weaker than Ore's condition, because $n + \kappa - \alpha + 1 \leq n$ holds in the case where $\alpha \geq \kappa + 1$.

Theorem C (*Fraisse and Jung [8]*). *Let G be a graph. If $\sigma_2 \geq n + \kappa - \alpha + 1$, then G is Hamiltonian.*

Bauer, Broersma, Li and Veldman [2] gave a σ_3 condition concerning the order and the connectivity.

Theorem D (*Bauer et al. [2]*). *Let G be a 2-connected graph. If $\sigma_3 \geq n + \kappa$, then G is Hamiltonian.*

Ozeki and the third author [15] obtained a σ_4 condition concerning the order, the connectivity and the independence number.

Theorem E (*Ozeki and Yamashita [15]*). *Let G be a 3-connected graph. If $\sigma_4 \geq n + \kappa + \alpha - 1$, then G is Hamiltonian.*

The degree sum conditions of Theorems C–E are best possible in a sense. In [15], by comparing them, that is,

Theorem C: $\sigma_2 \geq n + \kappa - \alpha + 1$,
Theorem D: $\sigma_3 \geq n + \kappa$,
Theorem E: $\sigma_4 \geq n + \kappa + \alpha - 1$.

Ozeki and the third author pointed out that the difference between the sharp lower bound of the σ_2 condition and that of the σ_3 condition is $\alpha - 1$, and the deference between that of the σ_3 condition and that of the σ_4 condition is also $\alpha - 1$. Furthermore, they showed that the same relation holds for the cyclability.

The cyclability versions of Theorems B–E are obtained by Broersma, H. Li, J. Li, Tian and Veldman (Theorem F(1) and (3)) and Ozeki and the third author (Theorem F(2) and (4)).

Theorem F (*Broersma et al. [5], Ozeki and Yamashita [15]*). *Let G be a graph and let $S \subseteq V(G)$ with $\kappa(S) \geq 2$.*

(1) *If $\alpha(S) \leq \kappa(S)$, then S is cyclable in G.*
(2) *If $\sigma_2(S) \geq n + \kappa(S) - \alpha(S) + 1$, then S is cyclable in G.*
(3) *If $\sigma_3(S) \geq n + \kappa(S)$, then S is cyclable in G.*
(4) *If $\sigma_4(S) \geq n + \kappa(S) + \alpha(S) - 1$ and $\kappa(S) \geq 3$, then S is cyclable in G.*

Motivated by the relations on the hamiltonicity and the cyclability, they posed the following conjecture.

Conjecture G (*Ozeki and Yamashita [15]*). *Let G be a graph, and let $S \subseteq V(G)$ with $\kappa(S) \geq k \geq 4$. If $\sigma_{k+1}(S) \geq n + \kappa(S) + (k-2)(\alpha(S) - 1)$, then S is cyclable in G.*

We now know the same relation as the hamiltonicity holds for the cyclability. Therefore we expect that the same relation holds for the circumference.

2. Main theorems

In this section, we investigate degree sum conditions for the circumference and obtain a new σ_4 condition. We denote by $c(G)$ the circumference of a graph G. Bermond [3] and Linial [11], independently, gave an Ore's type condition.

Theorem H (*Bermond [3], Linial [11]*). *Let G be a 2-connected graph. Then $c(G) \geq \min\{\sigma_2, n\}$.*

Like the hamiltonicity, from Theorem B, we should consider degree conditions for graphs satisfying $\alpha \geq \kappa + 1$. Hence we expect that the statement corresponding to Theorems C and F(2) holds. However, the statement "If G is a 2-connected graph, then $c(G) \geq \min\{\sigma_2 - \kappa + \alpha - 1, n\}$" does not hold. We consider a graph $G := K_k + mK_1$, where k and m are integers such that $k \geq 2$ and $m \geq k + 2$. Then $\sigma_2 = 2k$, $\kappa = k$ and $\alpha = m$. Therefore $c(G) = 2k < k + m - 1 = \sigma_2 - \kappa + \alpha - 1$.

One might be concerned whether the statement corresponding to Theorems D and F(3) holds or not. The third author [16] showed that it holds.