Discrete Mathematics 333 (2014) 101–105

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Vertex arboricity of toroidal graphs with a forbidden cycle

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ARTICLE INFO

Article history:
Received 7 November 2013
Received in revised form 10 June 2014
Accepted 11 June 2014
Available online 1 July 2014

Keywords:
Vertex arboricity
Toroidal graphs
Discharging

ABSTRACT

The vertex arboricity $a(G)$ of a graph G is the minimum k such that $V(G)$ can be partitioned into k sets where each set induces a forest. For a planar graph G, it is known that $a(G) \leq 3$. In two recent papers, it was proved that planar graphs without k-cycles for some $k \in \{3, 4, 5, 6, 7\}$ have vertex arboricity at most 2. For a toroidal graph G, it is known that $a(G) \leq 4$. Let us consider the following question: do toroidal graphs without k-cycles have vertex arboricity at most 2? It was known that the question is true for $k = 3$, and recently, Zhang proved the question is true for $k = 5$. Since a complete graph on 5 vertices is a toroidal graph without any k-cycles for $k \geq 6$ and has vertex arboricity at least three, the only unknown case was $k = 4$. We solve this case in the affirmative; namely, we show that toroidal graphs without 4-cycles have vertex arboricity at most 2.

1. Introduction

Let $[n] = \{1, \ldots, n\}$. Only finite, simple graphs are considered. Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. The *vertex arboricity* of a graph G, denoted $a(G)$, is the minimum k such that $V(G)$ can be partitioned into k sets $V_1, \ldots, V_k$ where $G[V_i]$ is a forest for each $i \in [k]$. This can be viewed as a vertex coloring f with k colors where each color class V_i induces a forest; namely, $G[f^{-1}(i)]$ is an acyclic graph for each $i \in [k]$. The *girth* of a graph G is the length of the smallest cycle in G. Note that a graph with no cycles is a forest, and it has vertex arboricity 1.

Vertex arboricity, also known as point arboricity, was first introduced by Chartrand, Kronk, and Wall [2] in 1968. Among other things, they proved Theorem 1.1. Shortly after, Chartrand and Kronk [1] showed that Theorem 1.1 is sharp by constructing a planar graph with vertex arboricity 3, and they also proved Theorem 1.2.

Theorem 1.1 (*[2]*)**.** *If G is a planar graph, then $a(G) \leq 3$.*

Theorem 1.2 (*[1]*)**.** *If G is an outerplanar graph, then $a(G) \leq 2$.*

We direct the readers to the work of Stein [10] and Hakimi and Schmeichel [5] for a complete characterization of maximal plane graphs with vertex arboricity 2.

In 2008, Raspaud and Wang [9] not only determined the order of the smallest planar graph G with $a(G) = 3$, but also found several sufficient conditions for a planar graph to have vertex arboricity at most 2 in terms of forbidden small structures; namely, they proved that a planar graph with either no triangles at distance less than 2 or no k-cycles for some fixed $k \in \{3, 4, 5, 6\}$ has vertex arboricity at most 2. Chen, Raspaud, and Wang [3] showed that forbidding intersecting triangles is also sufficient for planar graphs. In [9], Raspaud and Wang asked the following question:

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http://dx.doi.org/10.1016/j.disc.2014.06.011
0012-365X/

Question 1.3 ([9]). *What is the maximum integer μ where for all $k \in \{3, \dots, \mu\}$, a planar graph G with no k-cycles has $a(G) \leq 2$?*

Raspaud and Wang's results imply $6 \leq \mu \leq 21$. The lower bound was increased to 7 by Huang, Shiu, and Wang [6] since they proved planar graphs without 7-cycles have vertex arboricity at most 2.

We completely answer the question for toroidal graphs, which are graphs that are embeddable on a torus with no crossings.

Kronk [7] and Cook [4] investigated vertex arboricity on higher surfaces in 1969 and 1974, respectively.

Theorem 1.4 ([7]). *If G is a graph embeddable on a surface of positive genus g, then $a(G) \leq \lfloor \frac{9+\sqrt{1+48g}}{4} \rfloor$.*

Theorem 1.5 ([4]). *If G is a graph embeddable on a surface of genus g with no 3-cycles, then $a(G) \leq 2 + \sqrt{g}$.*

Theorem 1.6 ([4]). *If G is a graph embeddable on a surface of positive genus g with girth at least $5 + 4\log_3 g$, then $a(G) \leq 2$.*

Theorem 1.4 says every toroidal graph G has $a(G) \leq 4$. Theorem 1.5 says a toroidal graph with no 3-cycles has vertex arboricity at most 3, and Theorem 1.6 only guarantees that toroidal graphs with girth at least 5 have vertex arboricity at most 2. Both of these cases were improved by Kronk and Mitchem [8] who showed Theorem 1.7. Recently, Zhang [11] showed Theorem 1.8, which says that forbidding 5-cycles in toroidal graphs is sufficient to guarantee vertex arboricity at most 2.

Theorem 1.7 ([8]). *If G is a toroidal graph with no 3-cycles, then $a(G) \leq 2$.*

Theorem 1.8 ([11]). *If G is a toroidal graph with no 5-cycles, then $a(G) \leq 2$.*

Since the complete graph on 5 vertices is a toroidal graph with no cycles of length at least 6 and has vertex arboricity 3, the only remaining case is when 4-cycles are forbidden in toroidal graphs; this is our main result.

Theorem 1.9. *If G is a toroidal graph with no 4-cycles, then $a(G) \leq 2$.*

In Section 2, we will prove some structural lemmas needed in Section 3, where we prove Theorem 1.9 using (simple) discharging rules. Note that Theorem 1.9 implies that every planar graph without 4-cycles have vertex arboricity at most 2, which is a result in [9].

2. Lemmas

From now on, let G be a counterexample to Theorem 1.9 with the fewest number of vertices. It is easy to see that G must be 2-connected and the minimum degree of a vertex of G is at least 4.

A graph is k-regular if every vertex in the graph has degree k. A set $S \subseteq V(G)$ of vertices is k-regular if every vertex in S has degree k in G. A *triangular cycle* is a cycle adjacent to a triangle. A (partial) 2-coloring f of G is *good* if each color class induces a forest.

Lemma 2.1. *If $V(G)$ contains a 4-regular set S where $G[S]$ is a cycle C, then every good coloring f of $G[V(G) \setminus S]$ that does not extend to all of G has either*

Case 1: $f(v)$ the same for every vertex $v \notin S$ that has a neighbor in S, or
Case 2: $f(x) \neq f(y)$ for all $v \in S$ such that $N(v) \setminus S = \{x, y\}$ and C is an odd cycle.

Proof. Let $S = \{v_1, \dots, v_s\}$ where $v_1, \dots, v_s$ are the vertices of C in this order. For each $i \in [s]$, let $N(v_i) \setminus S = \{x_i, y_i\}$. Obtain a good coloring f of $G[V(G) \setminus S]$ by the minimality of G. We will show that if f does not satisfy one of the two conditions in the statement, then f can be extended to all of G.

If s is even and $\{f(x_i), f(y_i)\} = \{1, 2\}$ for each $i \in [s]$, then let $f(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$ to extend f to all of G.

We know that there exists at least one index $j \in [s]$ with $f(x_j) = f(y_j)$ since we are not in Case 2. For each $i \in [s]$ with $f(x_i) = f(y_i)$, let $f(v_i) = \begin{cases} 1 & \text{if } f(x_i) = f(y_i) = 2 \\ 2 & \text{if } f(x_i) = f(y_i) = 1 \end{cases}$. Now, consider the vertices of C in cyclic order starting with $i = j$, and for $f(v_i)$ that is not defined yet, let $f(v_i) = \begin{cases} 1 & \text{if } f(v_{i-1}) = 2 \\ 2 & \text{if } f(v_{i-1}) = 1 \end{cases}$ for all i. We claim that this coloring f is now a good coloring of all of G, which is a contradiction.

Note that f cannot have a monochromatic cycle that only uses vertices of $V(G) \setminus S$. Also, f cannot have a monochromatic cycle where x_i, v_i, y_i are consecutive vertices on this cycle since $f(x_i) = f(v_i) = f(y_i)$ never happens. Moreover, f cannot have a monochromatic cycle where v_i, v_{i+1}, x_i are consecutive vertices on this cycle since $f(v_i) = f(v_{i+1})$ implies that $f(x_{i+1}) = f(y_{i+1}) \neq f(v_{i+1})$. Thus, a monochromatic cycle in f must be C itself, which is possible only in Case 1. □

Lemma 2.2. *$V(G)$ does not contain a 4-regular set S where $G[S]$ is a triangular cycle.*