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On monochromatic configurations for finite colorings[☆]

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ABSTRACT

Answering a question of Gurevich, Graham proved that, given any $\delta > 0$, for any finite coloring of the plane, there is a triangle of area δ having all of its three vertices of the same color. Questions were asked about similar results for parallelograms, rhombuses etc. For any coloring of the plane, a trapezoid is called *monochromatic* if its four vertices have the same color. In this paper, we prove that, for any $\delta > 0$ and any finite coloring of the plane, there exist infinitely many monochromatic trapezoids of area $\delta > 0$ that are translates of the same trapezoid. We shall have some related results for triangles.

1. Introduction

Before we start discussing the problem considered in this paper, we briefly describe the related area of Euclidean Ramsey Theory which was developed in the pioneering papers [6–8] of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus. Here, one considers the problem of determining the finite geometric configurations F in the d-dimensional Euclidean space $\mathbb{R}^d$ which have the property that given a positive integer r, every r-coloring of $\mathbb{R}^k$ for some $k \geq d$ yields a monochromatic copy (here by a copy one means the image under some element in the group of Euclidean motions of $\mathbb{R}^k$) of the configuration F. A finite configuration F having the above property is said to be *Ramsey*. However, in this research, the dimension of the partitioned space is typically larger, than the dimension of the given configuration F. For an introduction to this area we refer to Chapter 7 of the monograph [11] or Chapter 5 of [2]. In connection to the problems taken up in this paper, we mention that I. Křı̂ž [13] established that all trapezoids are Ramsey. That is, given a set F of four points in some Euclidean space forming the vertices of a trapezoid, there is some suitably large k such that for every finite coloring of $\mathbb{R}^k$, there is a monochromatic copy of F; this generalizes a previous result of Frankl and Rödl [9] for triangles.

However, the questions considered here are about getting certain monochromatic configurations in a fixed dimension.

In the 1970s, Gurevich asked whether for any finite coloring of the plane, there always exists a triangle of unit area such that all of its vertices have the same color. A triangle having all of its vertices of the same color is simply called a *monochromatic triangle* (for instance, see Graham [11, Chapter 7]) and similarly for other configurations.

In 1980, Graham [10] answered this question positively. In fact, Graham [10,11] proved the following stronger result: *given any $\delta > 0$, for any finite coloring of the plane and any pair of intersecting lines, there exists a monochromatic triangle of area δ having two sides parallel to the given lines.* A short proof of the above result was later given by the first author [1] exploiting the main idea of Graham. Recently, a nice short proof of Gurevich's conjecture has been given by Dumitrescu and

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Jiang [5] with a different technique. Graham's proof yields a straightforward generalization involving vertices of a higher dimensional simplex. This has been indicated in Graham [11] where questions were asked about similar results for parallelograms, rhombuses, etc. For any coloring of the plane, a trapezoid is called *monochromatic* if its four vertices have the same color. A similar result for trapezoids was considered by the first author and Rath in [3]. Recently, the proof of the main result in [3] has been reported as incorrect in [4], where a proof of the following weaker result has been given.

For a positive integer r, given any r-coloring, we are ensured of a monochromatic trapezoid with two sides parallel to the x-axis and having an area $\frac{1}{2}(i+j)r!$ where $2 \le i+j \le 2r$.

However, if one does not insist that two sides of the quadrilateral should be parallel, then it has been proved in the above paper [4] that *given any $\delta > 0$, for any finite coloring of the plane, there is always a quadrilateral of area δ such that one of its diagonals divides it into two triangles of equal area and all of its four vertices are of the same color.*

In this paper, the following results are proved:

Theorem 1. *For any $\delta > 0$ and any finite coloring of the plane, there exist infinitely many monochromatic trapezoids of area δ which are translates of the same trapezoid with two sides parallel to the x-axis.*

We shall give a similar result for triangles. To describe these, we introduce the notion of l-triangles. A triangle ABC where the side BC is divided into l equal parts will be called an *l-triangle*. For convenience, all of its vertices A, B, C together with the $(l-1)$ points on BC dividing it into equal parts are called *the vertices of the l-triangle ABC*. An l-triangle is called a *monochromatic l-triangle* if all of its vertices have the same color.

Theorem 2. *Given any positive integer l, for any $\delta > 0$ and any finite coloring of the plane, there exist infinitely many monochromatic l-triangles of area δ which are translates of the same l-triangle.*

Remark 1. From the proof of Theorem 1 (resp. Theorem 2), we shall observe that, the vertices of monochromatic trapezoids (resp. l-triangles) are translates of the same trapezoid (resp. l-triangle) with all of its vertices in $\sqrt{\delta}A_r$ (resp. $A_{r,l}$), where A_r and $A_{r,l}$ are as in Corollaries 1 and 2, respectively.

We would like to pose the following problem here.

Problem 1. For any $\delta > 0$ and any finite coloring of the plane, is there a monochromatic isosceles trapezoid of area δ? If the answer is affirmative, are there infinitely many congruent monochromatic isosceles trapezoids of area δ?

While the main ingredient in our proofs is the celebrated van der Waerden theorem, our methods involve various induced colorings. For instance, given a finite coloring on the integer points, one can color a subset of points (here often a row of points) by nonempty subsets of the set of colors which appear in a particular subset. In general, an induced finite coloring on a row of points can be a set of finitely many parameters each taking finitely many values depending on the given coloring. In order to apply van der Waerden's theorem, one looks at horizontal or vertical shifts of these blocks of integer points. Theorems 3 and 4, in Sections 2 and 3, respectively, involve coloring of the integer points, by suitable scaling, the main results are obtained for coloring of all the points in the plane. The proof of Theorem 4 is based on the technique of Dumitrescu and Jiang [5], with a fairly straightforward adaptation that replaces colorings of lines by colorings of line segments (as sets of integer points) to ensure that the triangle on the integer points has bounded dimensions. The proof of Theorem 3 is a more sophisticated application of the same technique with some additional ideas, and a trick that cancels two equal lengths while adding the lengths of the parallel lines of the trapezoid involved.

2. Proof of Theorem 1

In order to prove Theorem 1, we prove the following stronger theorem on the integer points; Theorem 1 will be deduced from it by suitable scaling.

Theorem 3. *Given any positive integer r, there exist two positive integers M_r and N_r such that, for any r coloring of sets of all integer points in the plane, there is a monochromatic trapezoid of area M_r with two sides parallel to the x-axis and all of its vertices being integer points in the square with the diagonal vertices $(0, 0)$ and (N_r, N_r).*

Proof. We will employ the well known van der Waerden theorem ([14], one may also see [12]):

Given positive integers k and r, there exists a positive integer $W = W(k, r)$ such that for any r coloring of the set $\{1, 2, \dots, W\}$, there is a monochromatic arithmetic progression of length k.

For a positive integer r, writing $R = r^2(r+1)/2$ and $W = W(R!+1, 2^R-1)$, we define

$$N_r := W!r^2(r+1)^2 + r,$$

$$M_r := \frac{1}{2}R!W!(r+1).$$

Let an r coloring of the set of all integer points in the plane be given and $\{C_1, C_2, \dots, C_r\}$ be the set of colors. Now, for integers i, j, consider the set

$$L_{i,j} := \{(i(r+1)+n, j) : n = 0, 1, \dots, r\},$$

a set consisting of $r+1$ consecutive points on a horizontal line.