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Realizing degree sequences as Z_3-connected graphs

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ABSTRACT

An integer-valued sequence $\pi = (d_1, \dots, d_n)$ is *graphic* if there is a simple graph G with degree sequence of π. We say the π has a realization G. Let Z_3 be a cyclic group of order three. A graph G is Z_3-*connected* if for every mapping $b : V(G) \to Z_3$ such that $\sum_{v\in V(G)} b(v) = 0$, there is an orientation of G and a mapping $f : E(G) \to Z_3 - \{0\}$ such that for each vertex $v \in V(G)$, the sum of the values of f on all the edges leaving from v minus the sum of the values of f on the all edges coming to v is equal to $b(v)$. If an integer-valued sequence π has a realization G which is Z_3-connected, then π has a Z_3-*connected realization* G. Let $\pi = (d_1, \dots, d_n)$ be a nonincreasing graphic sequence with $d_n \ge 3$. We prove in this paper that if $d_1 \ge n - 3$, then π has a Z_3-connected realization unless the sequence is $(n-3, 3^{n-1})$ or is $(k, 3^k)$ or $(k^2, 3^{k-1})$ where $k = n - 1$ and n is even; if $d_{n-5} \ge 4$, then π has a Z_3-connected realization unless the sequence is $(5^2, 3^4)$ or $(5, 3^5)$.

1. Introduction

Graphs here are finite, and may have multiple edges without loops. We follow the notation and terminology in [2] except otherwise stated.

For a given orientation of a graph G, if an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then u is the *tail* of e and v is the *head* of e. For a vertex $v \in V(G)$, let $E^+(v)$ and $E^-(v)$ denote the sets of all edges having tail v or head v, respectively. A graph G is k-*flowable* if all the edges of G can be oriented and assigned nonzero numbers with absolute value less than k so that for every vertex $v \in V(G)$, the sum of the values on all the edges in $E^+(v)$ equals that of the values of all the edges in $E^-(v)$. If G is k-flowable we also say that G admits a nowhere-zero k-flow.

Let A be an abelian group with identity 0, and let $A^* = A - \{0\}$. Given an orientation and a mapping $f : E(G) \to A$, the *boundary* of f is a function $\partial f : V(G) \to A$ defined by, for each vertex $v \in V(G)$,

$$\partial f(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A mapping $b : V(G) \to A$ is a *zero-sum function* if $\sum_{v\in V(G)} b(v) = 0$. A graph G is A-*connected* if for every zero-sum function $b : V(G) \to A$, there exist an orientation of G and a mapping $f : E(G) \to A^*$ such that $\partial f(v) = b(v)$ for each $v \in V(G)$.

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The concept of k-flowability was first introduced by Tutte [19], and this theory provides an interesting way to investigate the coloring of planar graphs in the sense that Tutte [19] proved a classical theorem: a planar graph is k-colorable if and only if it is k-flowable. Jaeger et al. [10] successfully generalized nowhere-zero flow problems to group connectivity. The purpose of study in group connectivity is to characterize contractible configurations for integer flow problems. Let Z_3 be a cyclic group of order three. Obviously, if G is Z_3-connected, then G is 3-flowable.

An integer-valued sequence $\pi = (d_1, \ldots, d_n)$ is *graphic* if there is a simple graph G with degree sequence π. We say π has a *realization* G, and we also say G is a realization of π. If an integer-valued sequence π has a realization G which is A-connected, then we say that G is a A-connected realization of π for an abelian group A. In particular, if $A = Z_3$, then we say G is a Z_3-connected realization (or π has a Z_3-connected realization G). In this paper, we write every degree sequence $(d_1, \ldots, d_n)$ is in nonincreasing order. For simplicity, we use exponents to denote degree multiplicities, for example, we write $(6, 5, 4^4, 3)$ for $(6, 5, 4, 4, 4, 4, 3)$.

The problem of realizing degree sequences by graphs that have nowhere-zero flows or are A-connected, where A is an abelian group, has been studied. Luo et al. [17] proved that every bipartite graphic sequence with least element at least 2 has a 4-flowable realization. As a corollary, they confirmed the simultaneous edge-coloring conjecture of Cameron [3]. Fan et al. [6] proved that every degree sequence with least element at least 2 has a realization which contains a spanning Eulerian subgraph; such graphs are 4-flowable. Let A be an abelian group with $|A| = 4$. For a nonincreasing n-element graphic sequence π with least element at least 2 and sum at least $3n - 3$, Luo et al. [15] proved that π has a realization that is A-connected. Yin and Guo [20] determined the smallest degree sum that yields graphic sequences with a Z_3-connected realization. For the literature for this topic, the readers can see a survey [13]. In particular, Luo et al. [16] completely answered the question of Archdeacon [1]: Characterize all graphic sequences π realizable by a 3-flowable graph. The natural group connectivity version of Archdeacon's problem is as follows.

Problem 1.1. Characterize all graphic sequences π realizable by a Z_3-connected graph.

On this problem, Luo et al. [16] obtained the next two results.

Theorem 1.2. *Every nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_1 = n - 1$ and $d_n \geq 3$ has a Z_3-connected realization unless n is even and the sequence is $(k, 3^k)$ or $(k^2, 3^{k-1})$, where $k = n - 1$.*

Theorem 1.3. *Every nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_n \geq 3$ and $d_{n-3} \geq 4$ has a Z_3-connected realization.*

Motivated by Problem 1.1 and the results above, we present the following two theorems in this paper. These results extend the results of [16] by extending the characterizations to a large set of sequences.

Theorem 1.4. *A nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_1 \geq n - 3$ and $d_n \geq 3$ has a Z_3-connected realization unless the sequence is $(n - 3, 3^{n-1})$ for any n or is $(k, 3^k)$ or $(k^2, 3^{k-1})$, where $k = n - 1$ and n is even.*

Theorem 1.5. *A nonincreasing graphic sequence $(d_1, \ldots, d_n)$ with $d_n \geq 3$ and $d_{n-5} \geq 4$ has a Z_3-connected realization unless the sequence is $(5^2, 3^4)$ or $(5, 3^5)$.*

We end this section with some notation and terminology. A graph is trivial if $E(G) = \emptyset$ and nontrivial otherwise. A *k-vertex* denotes a vertex of degree k. Let P_n denote the path on n vertices and we call P_n a n-path. An *n-cycle* is a cycle on n vertices. The *wheel* W_k is the graph obtained from a k-cycle by adding a new vertex, the center of the wheel, and joining it to every vertex of the k-cycle. A wheel W_k is an *odd (even)* wheel if k is odd (even). For simplicity, we say W_1 is a triangle. For a graph G and $X \subseteq V(G)$, denote by $G[X]$ the subgraph of G induced by X. For two vertex-disjoint subsets V_1, V_2 of $V(G)$, denote by $e(V_1, V_2)$ the number of edges with one endpoint in V_1 and the other endpoint in V_2.

We organize this paper as follows. In Section 2, we state some results and establish some lemmas that will be used in the following proofs. We will deal with some special degree sequences, each of which has a Z_3-connected realization in Section 3. In Sections 4 and 5, we will give the proofs of Theorems 1.4 and 1.5.

2. Lemmas

Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_1 \geq \cdots \geq d_n$. Throughout this paper, we use $\bar{\pi}$ to represent the sequence $(d_1 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1})$, which is called the *residual sequence* obtained from π by deleting d_n. The following well-known result is due to Hakimi [8,9] and Kleitman and Wang [11].

Theorem 2.1. *A graphic sequence has even sum. Furthermore, a sequence π is graphic if and only if $\bar{\pi}$ is graphic.*

Some results in [4,5,7,10,12] on group connectivity are summarized as follows.

Lemma 2.2. *Let A be an abelian group with $|A| \geq 3$. The following results are known:*

(1) *K_1 is A-connected;*
(2) *K_n and K_n^- are A-connected if $n \geq 5$;*
(3) *An n-cycle is A-connected if and only if $|A| \geq n + 1$;*