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Note

On the total coloring of generalized Petersen graphs[☆]

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A B S T R A C T

We show that "almost all" generalized Petersen graphs have total chromatic number 4. More precisely: for each integer $k \geq 2$, there exists an integer $N(k)$ such that, for any $n \geq N(k)$, the generalized Petersen graph $G(n, k)$ has total chromatic number 4.

1. Introduction

The Total Coloring Conjecture states that the total chromatic number of any graph is at most $\Delta + 2$, where Δ is the maximum degree of a graph [1,18]. This conjecture has been proved for cubic graphs, so the total chromatic number of a cubic graph is either 4 (called Type 1) or 5 (called Type 2) [14,17], see also [5] for a recent concise proof. It is NP-hard to decide whether a cubic graph is Type 1, even restricted to bipartite cubic graphs [16].

The smallest Type 2 cubic graph is K_4 and the smallest Type 2 bipartite cubic graph is $K_{3,3}$. The Type of all cubic graphs with order up to 16 is established [3,9] as well as the Type of infinite families of cubic graphs [3,7,11]. So far every known Type 2 cubic graph contains a square or a triangle, and computational results show that a possible Type 2 cubic graph with girth greater than 4 must have at least 34 vertices [2].

Furthermore, recent results on the fractional total chromatic number support the evidence that the girth of a graph is a relevant parameter in the study of total coloring: in particular, it is proved in [10] that if the girth of a cubic graph is sufficiently large then its fractional total chromatic number is 4. The facts listed above lead us to consider the following question:

Question 1 (*Brinkmann et al. [2]*). *Does there exist a Type 2 cubic graph with girth greater than 4?*

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In this paper, we consider the Type of a well-studied class of cubic graphs, called generalized Petersen graphs [19,13,4] and denoted $G(n,k)$. It is proved in [3] that $G(n,1)$ graphs are all Type 1 but $G(5,1)$. It is easy to verify that for $2 \le k \le \frac{n}{2}$, $G(n,k)$ has girth greater than 4 whenever $n \notin \{2k, 3k, 4k\}$; so one could hope to find an answer to Question 1 among this family. However, we show here that, for each integer $k \ge 2$, there exists an integer $N(k)$ such that, for any $n \ge N(k)$, the generalized Petersen graph $G(n,k)$ is Type 1. So, for $k \ge 2$, a Type 2 $G(n,k)$ may exist only for $n < N(k)$. This may happen: $G(9,3)$ is Type 2 [2]. Nevertheless, our results combined with a computer search [6] show that $G(9,3)$ is the only Type 2 graph among generalized Petersen graphs $G(n,k)$, for $2 \le k \le 6$.

2. Preliminaries

A *semi-graph* is a triple $G = (V, E, S)$ where V is the set of vertices of G, E is a set of edges having two distinct endpoints in V, and S is a set of *semi-edges* having one endpoint in V (the notion of semi-graph is similar to the one of "multipole" used by other authors [8,12]). We denote an edge having endpoints v and w by vw and a semi-edge having endpoint v as $v\cdot$. When vertex v is an endpoint of $e \in E \cup S$ we say that v and e are *incident*. Two elements of $E \cup S$ incident to the same vertex, or two vertices incident to the same edge, are called *adjacent*.

In this work, we are mainly interested in graphs and semigraphs such that there are exactly three elements (edges and/or semi-edges) incident to every vertex. These are called *cubic graphs* and *cubic semi-graphs*, respectively.

For $k \in \mathbb{N}$, a *k-vertex-coloring* of G is a map $C^V \colon V \to \{1, 2, \ldots, k\}$, such that $C^V(x) \neq C^V(y)$ whenever x and y are two adjacent vertices. Similarly, a *k-edge-coloring* of G is a map $C^E \colon E \cup S \to \{1, 2, \ldots, k\}$, such that $C^E(e) \neq C^E(f)$ whenever e and f are adjacent elements of $E \cup S$. A *k-total-coloring* of G is a map $C^T \colon V \cup E \cup S \to \{1, 2, \ldots, k\}$, such that $C^T|_V$ is a vertex-coloring, $C^T|_{E \cup S}$ is an edge-coloring, and $C^T(e) \neq C^T(v)$ whenever $e \in E \cup S$, $v \in V$, and e is incident to v.

Definition 1. A proper 4-edge-coloring C^E of a cubic semi-graph $G = (V, E, S)$ is called **strong 4-edge-coloring** if, for each edge $vw \in E$, we have $|\{C^E(e) | e \in E \cup S, e \text{ incident to } v \text{ or } w\}| = 4$.

Lemma 1 (*Brinkmann et al. [2]*). *Let $G = (V, E, S)$ be a cubic semi-graph. Each strong 4-edge-coloring C^E of G can be extended to a 4-total-coloring C^T with $C^T|_{E \cup S} = C^E$ and, for each 4-total-coloring C^T of G, $C^T|_{E \cup S}$ is a strong 4-edge-coloring.*

Lemma 1 implies that there exists a 4-total-coloring C^T of G if and only if there exists a strong 4-edge-coloring C^E of G. Furthermore, a strong 4-edge-coloring has the property that if we assign to each vertex v the color c, which is not used for the three elements incident to v, we produce a 4-total-coloring of G. In what follows, we say that c is *the color induced on v* by the strong 4-edge-coloring.

The next section is devoted to results on total colorings of generalized Petersen graphs, a well-known class of cubic graphs introduced by Watkins [19]. Following Watkins' notation, the generalized Petersen graph $G(n,k)$, $n \ge 3$ and $1 \le k \le n-1$, is the graph with vertex-set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, v_{n-1}\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : 0 \le i \le n-1\}$, with subscripts taken modulo n. Thus each $G(n,k)$ is cubic and $G(5,2)$ is the Petersen graph. Clearly, the graph $G(n,k)$ and the graph $G(n, n-k)$ are isomorphic and generalized Petersen graphs are usually defined for $k < \frac{n}{2}$. Here, we consider k into the entire interval $[1, n-1]$ in order to avoid boring specifications along the rest of the paper and, in the case $k = \frac{n}{2}$, we allow two edges between v_i and v_{i+k}.

3. Main results

In this section, we prove that "almost all" generalized Petersen graphs have total chromatic number 4. In order to prove our main theorem, we need to define the following semi-graph, which we denote by $F_{l,k}$, for $l \ge 2k-1$:

- the vertices of $F_{l,k}$ are $u_1, u_2, \ldots, u_l, v_1, v_2, \ldots, v_l$;
- the edges of $F_{l,k}$ are $u_i u_{i+1}$ for $1 \le i < l$, $u_i v_i$ for $1 \le i \le l$, $v_i v_{i+k}$ for $1 \le i \le l-k$;
- the semi-edges of $F_{l,k}$ are divided in two classes, left semi-edges and right semi-edges. Each class contains $k+1$ semi-edges numbered from 0 to k: the 0*th left semi-edge* is $u_1\cdot$; the *ith left semi-edge* is $v_i\cdot$, for $1 \le i \le k$; the 0*th right semi-edge* is $u_l\cdot$; and the $(i-l+k)$*th right semi-edge* is $v_i\cdot$, for $l-k+1 \le i \le l$.

Any semi-graph isomorphic to $F_{l,k}$ is called a *k-frieze* of length l. Fig. 3 presents the two 3-friezes $F_{6,3}$ and $F_{5,3}$, and the generalized Petersen graph $G(11,3)$.

Given two semi-edges $x\cdot$ and $y\cdot$, the *junction* of $x\cdot$ and $y\cdot$ means replacing $x\cdot$ and $y\cdot$ by an edge xy. We define the *merge* of a k-frieze F of length l and a k-frieze F' of length l' as the k-frieze FF' of length $l+l'$ obtained by the junction of the ith right semi-edge of F with the ith left semi-edge of F' for $0 \le i \le k$. The left semi-edges of FF' are those of F and the right ones are those of F'. We define the *closure* of a k-frieze F as the graph obtained by the junction, for each $0 \le i \le k$, of the ith left semi-edge of F with the ith right semi-edge of F itself. It is easy to check that the closure of a k-frieze of length $l > 2k$ is the generalized Petersen graph $G(l,k)$.

Given two strong 4-edge-colorings ϕ and ϕ' of k-friezes $F = F_{l,k}$ and $F' = F_{l',k}$ respectively, we say that ϕ is *compatible* with ϕ' if for each i from 0 to k: the color given by ϕ to the right ith semi-edge of F is equal to the color given by ϕ' to the left ith semi-edge of F'; and the color induced by ϕ on the end-vertex of the right ith semi-edge of F is distinct from the color induced by ϕ' on the end-vertex of the left ith semi-edge of F'. Then ϕ and ϕ' provide a strong 4-edge-coloring of the merge of F and F'.