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# On a general class of non-squashing partitions

Amanda Folsom[a,b,*], Youkow Homma[a], Jun Hwan Ryu[a], Benjamin Tong[a]

[a] *Yale University, Department of Mathematics, P.O. Box 208283, New Haven, CT 06520-8283, United States*

[b] *Amherst College, Department of Mathematics, Amherst, MA 01002, United States*

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ABSTRACT

We define $M$-sequence non-squashing partitions, which specialize to $m$-ary partitions (studied by Andrews, Churchhouse, Erdös, Hirschhorn, Knuth, Mahler, Rødseth, Sellers, and Sloane, among others), factorial partitions, and numerous other general partition families of interest. We establish an exact formula, various combinatorial interpretations, as well as the asymptotic growth of $M$-sequence non-squashing partition functions, functions whose associated generating functions are non-modular. In particular, we obtain an exact formula for the $m$-ary partition function, and by new methods, we recover Mahler's and Erdös' asymptotic for the $m$-ary partition function. We also establish new results on factorial partitions, colored $m$-ary partitions, and many other general families which have not been well understood or systematically studied. Finally, we conjecture Ramanujan-like congruences for the $M$-sequence non-squashing partition functions.



## 1. Introduction and statement of results

### 1.1. Introduction and motivation

Integer partitions have been a vibrant topic of study in number theory for almost three centuries. Among the many notable mathematicians who have studied integer partitions, Ramanujan made a tremendous impact in the subject by uncovering his celebrated congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \bmod 5,\\ p(7n+5) &\equiv 0 \bmod 7,\\ p(11n+6) &\equiv 0 \bmod 11, \end{aligned} \tag{1.1}$$

for the partition function $p(n) := \#\{\text{partitions of } n\}$, which hold for all $n \in \mathbb{N}_0$. Together, Hardy and Ramanujan further advanced the theory of partitions by introducing the "Circle Method" in analytic number theory, which led to their discovery of the asymptotic growth of the partition function. Namely, as $n \to \infty$, Hardy and Ramanujan showed that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right). \tag{1.2}$$

* Corresponding author at: Amherst College, Department of Mathematics, Amherst, MA 01002, United States.
*E-mail addresses:* amanda.folsom@yale.edu, afolsom@amherst.edu (A. Folsom), youkow.homma@yale.edu (Y. Homma), junhwan.ryu@yale.edu (J.H. Ryu), benjamin.tong@yale.edu (B. Tong).

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Hardy and Wright found an entire asymptotic expansion for $p(n)$, and later Rademacher and independently Selberg found an exact formula for $p(n)$ as an infinite series.

In addition to ordinary partitions, the so-called $m$-ary partitions, partitions into parts that are powers of some integer $m \geq 2$, have been a subject of study since the early 20th century. In 1969, Churchhouse [6] initiated the study of congruence properties for the $m$-ary partitions by stating the following conjecture on binary (2-ary) partitions. Here and throughout, we let $\beta_m(n) := \#\{m\text{-ary partitions of } n\}$.

**Conjecture 1.1** (*Churchhouse [6]*). *For all integers $k \geq 1$ and odd integers $n \geq 1$, we have that*

$$\beta_2(2^{2k+2}n) \equiv \beta_2(2^{2k}n) \bmod 2^{3k+2},$$
$$\beta_2(2^{2k+1}n) \equiv \beta_2(2^{2k-1}n) \bmod 2^{3k}.$$

Conjecture 1.1 was first proved by Rødseth [18]. Related work of Hirschhorn and Loxton [12] later determined all "admissible" congruences satisfied by the binary partition function. The Churchhouse conjectures were later extended to $m$-ary partitions ($m \geq 2$) by Andrews [2], Gupta [10], and Rødseth and Sellers [19]. In particular, more analogous to the Ramanujan congruences (1.1), Rødseth and Sellers proved the following congruence properties of the $m$-ary partition function on certain arithmetic progressions modulo powers of $m$. In what follows, the $\mathbb{N}$-valued function $c_r = c_{r,m}$ equals $2^{r-1}$ if $m$ is even, and equals 1 if $m$ is odd. The function $\sigma_r = \sigma_{r,m} := \sigma_{r,M}(\epsilon)$ is as defined in (1.6) with $M = \{m, m, m, \ldots\}$.

**Theorem 1.2** (*Rødseth–Sellers [19]*). *For any integers $r, n \geq 1$, we have that*

$$\beta_m(m^{r+1}n - \sigma_r - m) \equiv 0 \bmod \frac{m^r}{c_r}.$$

Prior to the Churchhouse conjectures, in 1940, Mahler [16] determined the asymptotic growth of the $m$-ary partition function as $n \to \infty$, namely,

$$\beta_m(n) \sim \exp\left(\frac{(\ln n)^2}{2\ln m}\right), \tag{1.3}$$

which naturally grows more slowly than $p(n)$ in (1.2). As we shall see, the generating function for $m$-ary partitions is not a modular form, while it is well known that the generating function for $p(n)$ is essentially modular. Mahler used methods other than the Circle Method to determine his asymptotic (1.3). By still different analytic methods, De Bruijn [7] and Pennington [17] were able to improve and generalize "Mahler's problem" on the asymptotic growth of $\beta_m(n)$. Other mathematicians have also studied $m$-ary partition function asymptotics by more elementary methods, such as Knuth [15] in the case $m = 2$ and Erdös [9], who in the same paper nearly recovered the Hardy–Ramanujan asymptotic for $p(n)$ in (1.2).

In 2004, Hirschhorn and Sellers [13] studied $m$-ary partitions through an entirely new lens, as they related $m$-ary partitions to *$m$-non-squashing partitions*, defined as follows.

**Definition 1.3** (*Hirschhorn–Sellers [13]*). For any integer $m \geq 2$, an *$m$-non-squashing partition* of a positive integer $n$ is a set of positive integers, $\{p_1, \ldots, p_k\}$, such that

(1) $p_1 + \cdots + p_k = n$,
(2) $p_1 \leq p_2 \leq \cdots \leq p_k$, and
(3) $(m-1)(p_1 + \cdots + p_{j-1}) \leq p_j$ for $j = 2, \ldots, k$.

Hirschhorn and Sellers beautifully showed that for any integers $n \geq 0$ and $m \geq 2$ the number of $m$-ary partitions of $n$ is equal to the number of $m$-non-squashing partitions of $n$, that is,

$$\alpha_m(n) = \beta_m(n), \tag{1.4}$$

where we let $\alpha_m(n) := \#\{m\text{-non-squashing partitions of } n\}$. (Note. As usual, $\alpha_m(0) = \beta_m(0) := 1$.) The combinatorial interpretation of $m$-ary partitions due to Hirschhorn and Sellers given by (1.4), as well as subsequent elaborations by Sellers and Sloane [20] have led to new methods by which the $m$-ary partitions can be studied. Non-squashing partitions have been linked to other interesting problems as well, including the box stacking problem [20], which we will elaborate upon in Section 1.2.4, double binary number systems [8], and recursively self-conjugating partitions [14]. As noted by Andrews and Sellers [5], the study of partitions using factorial numbers as parts also served to motivate the study of $m$-non-squashing partitions and the box stacking problem; however, the presumed relationship ultimately proved to be false. We address this question on factorial partitions in more detail in the following sections.