Discrete Mathematics 339 (2016) 1507–1511

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Weight of edges in normal plane maps

O.V. Borodin [a], A.O. Ivanova [b,*]

[a] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

[b] Ammosov North-Eastern Federal University, Yakutsk 677000, Russia

ARTICLE INFO

Article history:
Received 26 April 2015
Received in revised form 20 December 2015
Accepted 21 December 2015
Available online 15 January 2016

Keywords:
Planar graph
Plane map
Structure properties
3-polytope
Weight

ABSTRACT

The weight $w(e)$ of an edge e in a normal plane map (NPM) is the degree-sum of its end-vertices. An edge $e = uv$ is an (i, j)-edge if $d(u) \leq i$ and $d(v) \leq j$. In 1940, Lebesgue proved that every NPM has a $(3, 11)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge, where 7 and 6 are best possible. In 1955, Kotzig proved that every 3-polytope has an edge e with $w(e) \leq 13$, which bound is sharp. Borodin (1987), answering Erdős' question, proved that every NPM has either a $(3, 10)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge.

A vertex is simplicial if it is completely surrounded by 3-faces. In 2010, Ferencová and Madaras conjectured (in different terms) that every 3-polytope without simplicial 3-vertices has an edge e with $w(e) \leq 12$.

The purpose of our note is to prove that every NPM has either a simplicial 3-vertex adjacent to a vertex degree at most 10, or $(3, 9)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge. In particular, this confirms the above mentioned conjecture by Ferencová and Madaras. Furthermore, we construct a 3-polytope showing that the above term $(3, 9)$ is best possible.

1. Introduction

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three.

The degree of a vertex or a face x, that is the number of edges incident with x (loops and cut-edges are counted twice in the degree of vertex and face, respectively) is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc. An edge uv is an *(i, j)-edge* if $d(u) \leq i$ and $d(v) \leq j$.

The *weight* $w(e)$ of an edge e in a normal plane map (NPM) is the degree-sum of its end-vertices. By $\delta(G)$ and $w(G)$ we denote the minimum vertex degree and the minimum weight of edges of a graph G, respectively. We will drop the argument when it is clear from context.

Already in 1904, Wernicke [17] proved that every NPM with $\delta = 5$ satisfies $w \leq 11$. In 1940, Lebesgue [16] proved that every NPM has a $(3, 11)$-edge, or $(4, 7)$-edge, or $(5, 6)$-edge, where 7 and 6 are best possible. In 1955, Kotzig [15] proved that every 3-connected planar graph satisfies $w \leq 13$, which bound is sharp.

In 1972, Erdős (see [12]) conjectured that Kotzig's bound $w \leq 13$ holds for all planar graphs with $\delta \geq 3$. Barnette (see [12]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [1].

* Corresponding author.
E-mail addresses: brdnoleg@math.nsc.ru (O.V. Borodin), shmgnanna@mail.ru (A.O. Ivanova).

http://dx.doi.org/10.1016/j.disc.2015.12.021

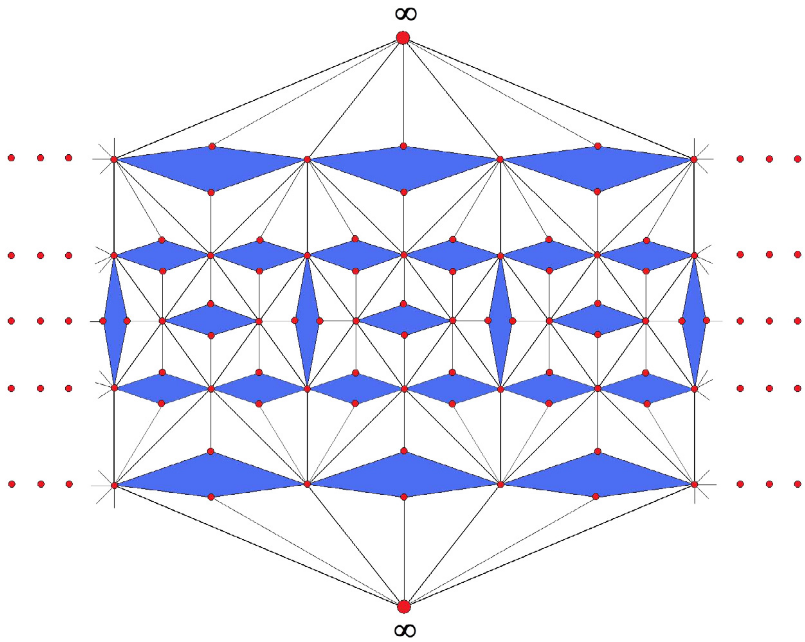

Fig. 1. A construction with no simplicial 3-vertices and $w(e) \geq 12$ for each edge *e*.

More generally, Borodin [2–4] proved that every NPM contains a (3, 10)-, or (4, 7)-, or (5, 6)-edge (as easy corollaries of some stronger structural facts having applications to coloring of plane graphs, see [5]).

Note that $\delta(K_{2,t}) = 2$ and $w(K_{2,t}) = t+2$, so w is unbounded if $\delta \leq 2$. An induced cycle $v_1 \dots v_{2k}$ in a graph is 2-*alternating* (Borodin [6]) if $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. This notion, along with its more sophisticated analogues (t-alternating subgraph, 3-alternator (Borodin, Kostochka, and Woodall [10]), cycle consisting of 3-paths (Borodin–Ivanova [8]), etc.), turns out to be useful for the study of graph coloring, since it sometimes provides crucial reducible configurations in coloring and partition problems (more often, on sparse plane graphs, see Borodin [6,7,5]). Its first application was to show that the total chromatic number of planar graphs with maximum degree Δ at least 14 equals $\Delta + 1$ (Borodin [6]). In particular, forbidding 2-alternating 4-cycles implies $w \leq 17$ (Borodin [1]), while forbidding all 2-alternating cycles implies $w_2 \leq 15$ (Borodin [6]), where both bounds are tight.

In some coloring applications, it is important to find a light edge incident with one or two 5^--faces. Nowadays, the maximum weight of edges is known for many interesting classes of plane graphs (further examples and references can be found in [3,2,4,5,9,13,14]).

A vertex is *simplicial* if it is completely surrounded by 3-faces. In 2010, Ferencová and Madaras [11] conjectured (in different terms) that every 3-connected plane graph without simplicial 3-vertices has an edge of weight at most 12.

The purpose of our note is to prove a refinement of the above mentioned result in Borodin [2–4], which, in particular, confirms the conjecture in Ferencová–Madaras [11].

Theorem 1. *Every normal plane map has either a simplicial* 3*-vertex adjacent to a vertex of degree at most* 10*, or* (3, 9)*-edge, or* (4, 7)*-edge, or* (5, 6)*-edge, where all bounds are sharp.*

Corollary 2. *Every* 3*-connected plane graph without simplicial* 3*-vertices has an edge of weight at most* 12.

2. Proving Theorem 1

The sharpness of the bound 13 in Theorem 1 follows by putting a 3-vertex into each face of the icosahedron; to attain 6, we put a 5-vertex into each face of the dodecahedron. To reach the bound 7, we take the (3, 4, 4, 4)-Archimedean solid, in which every vertex is incident with a 3-face and three 4-faces, and put a 4-vertex into each 4-face.

In Fig. 1, we see a 3-connected plane graph with neither simplicial 3-vertices nor vertices of degree from 4 to 8, which confirms that the term (3, 9) is best possible.

2.1. Discharging and its consequences

Suppose M' is a counterexample to Theorem 1 with the fewest vertices. Hence M' is connected. By M denote a counterexample to Theorem 1 with the most edges on the same vertices as M'. In other words, adding any diagonal to a 4^+-face of M must result in an edge claimed in Theorem 1.