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Waiter–Client and Client–Waiter planarity, colorability and minor games

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ABSTRACT

For a finite set X, a family of sets $\mathcal{F} \subseteq 2^X$ and a positive integer q, we consider two types of two player, perfect information games with no chance moves. In each round of the $(1 : q)$ Waiter–Client game $(X, \mathcal{F})$, the first player, called Waiter, offers the second player, called Client, $q + 1$ elements of the board X which have not been offered previously. Client then chooses one of these elements which he claims and the remaining q elements are claimed by Waiter. Waiter wins this game if by the time every element of X has been claimed by some player, Client has claimed all elements of some $A \in \mathcal{F}$; otherwise Client is the winner. Client–Waiter games are defined analogously, the main difference being that Client wins the game if he manages to claim all elements of some $A \in \mathcal{F}$ and Waiter wins otherwise. In this paper we study the Waiter–Client and Client–Waiter versions of the non-planarity, K_t-minor and non-k-colorability games. For each such game, we give a fairly precise estimate of the unique integer q at which the outcome of the game changes from Client's win to Waiter's win. We also discuss the relation between our results, random graphs, and the corresponding Maker–Breaker and Avoider–Enforcer games.

1. Introduction

The theory of positional games on graphs and hypergraphs goes back to the seminal papers of Hales and Jewett [20], of Lehman [31] and of Erdős and Selfridge [16]. It has since become a highly developed area of combinatorics (see, e.g., the monograph of Beck [3] and the recent monograph [23]). The most popular and widely studied positional games are the so-called Maker–Breaker games. Let p and q be positive integers, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. In each round of the biased $(p : q)$ Maker–Breaker game $(X, \mathcal{F})$, Maker claims p previously unclaimed elements of X and then Breaker responds by claiming q previously unclaimed elements of X. Maker wins this game if, by the time every element of X has been claimed, he has claimed all elements of some set $A \in \mathcal{F}$; otherwise Breaker is the winner. The set X is called the *board* of the game, the elements of $\mathcal{F}$ are called the *winning sets* and the integers p and q are Maker's bias and Breaker's bias, respectively. Since this is a finite, perfect information game with no chance moves and no possibility of a draw, one of the two players must have a winning strategy. Moreover, it is not hard to see that Maker–Breaker games are *bias monotone*, that is, claiming more board elements than his bias specifies per round cannot harm that player. In particular, there exists a unique positive integer $b_{\mathcal{F}}$ such that Breaker has a winning strategy for the $(1 : q)$ game $(X, \mathcal{F})$ if and only if $q \geq b_{\mathcal{F}}$; we refer to this integer as the *threshold bias* of the Maker–Breaker game $(X, \mathcal{F})$.

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The so-called Avoider–Enforcer games form another class of well-studied positional games. In such games, Enforcer aims to force Avoider to claim all elements of some set $A \in \mathcal{F}$. Avoider–Enforcer games are sometimes referred to as misère Maker–Breaker games. There are two different sets of rules for Avoider–Enforcer games: *strict rules* under which the number of board elements a player claims per round is precisely his bias and *monotone rules* under which the number of board elements a player claims per round is at least as large as his bias (for more information on Avoider–Enforcer games see, for example, [24,22,23]).

One major motivation for studying biased Maker–Breaker and Avoider–Enforcer games is their relation to the theory of random graphs via the so-called *probabilistic intuition*. Consider, for example, a $(1 : q)$ Maker–Breaker game $(X, \mathcal{F})$, where $X = E(K_n)$ is the edge-set of the complete graph on n vertices. The following heuristic argument, first employed by Chvátal and Erdős in [12], can be used to predict the winner of this game. This heuristic suggests that the player who has a higher chance to win the game when both players are playing randomly is also the one who wins the game when both players are playing optimally. More precisely, if the random graph $G(n, m)$ with $m = \lceil \binom{n}{2}/(q+1) \rceil$ edges contains all edges of some $A \in \mathcal{F}$ with probability tending to 1 as n tends to infinity, then Maker has a winning strategy for $(E(K_n), \mathcal{F})$. If, on the other hand, this probability tends to 0 as n tends to infinity, then $(E(K_n), \mathcal{F})$ is Breaker's win. This is highly unexpected as, in any positional game, both players have deterministic optimal strategies. Moreover, in most natural games, playing randomly against an optimal opponent leads to very poor results. As noted above, this is just a heuristic and does not always predict the outcome of the game correctly. Nevertheless, the probabilistic intuition is remarkably useful. Natural examples where this heuristic predicts the winner correctly, include the connectivity game (that is, $\mathcal{F}$ consists of all connected subgraphs of K_n) [19] and the Hamiltonicity game (that is, $\mathcal{F}$ consists of all Hamiltonian subgraphs of K_n) [29]. On the other hand, it was proved in [7] that the probabilistic intuition fails (though another probabilistic reasoning is in play here) for the H-game (that is, $\mathcal{F}$ consists of all copies of some fixed predetermined graph H in K_n). The probabilistic intuition is also useful when analyzing biased Avoider–Enforcer games (especially under monotone rules). Some examples can be found in [21] and [22].

In this paper, we study *Waiter–Client* and *Client–Waiter* positional games. Such games are closely related to Maker–Breaker and Avoider–Enforcer games; the main difference being the process of selecting board elements. In every round of the biased $(p : q)$ Waiter–Client game $(X, \mathcal{F})$, the first player, called Waiter, offers the second player, called Client, $p + q$ previously unclaimed elements of X. Client then chooses p of these elements which he claims, and the remaining q elements are claimed by Waiter. If, in the final round of the game, only $1 \le t < p + q$ unclaimed elements remain, then Client chooses $\max\{0, t - q\}$ elements which he claims and the remaining $\min\{t, q\}$ elements are claimed by Waiter. The game ends as soon as all elements of X have been claimed. Waiter wins this game if he manages to force Client to claim all elements of some $A \in \mathcal{F}$; otherwise Client is the winner. Client–Waiter games are defined analogously, the main difference being that Client wins if and only if he manages to claim all elements of some $A \in \mathcal{F}$ (otherwise Waiter is the winner). Additionally, there are two technical differences between Client–Waiter and Waiter–Client games. Firstly, in a Client–Waiter game, Waiter is allowed to offer less board elements per round than his bias specifies. More precisely, in every round of a $(p : q)$ Client–Waiter game, Waiter offers t elements, where $p \le t \le p + q$. Client chooses p of these, which he keeps, and the remaining $t - p$ elements are claimed by Waiter. Secondly, if there are $r < p + q$ free elements offered to Client in the final round of the game, he first claims $\min\{r, p\}$ of these and any remaining elements are claimed by Waiter. As with Maker–Breaker games, it is not hard to see that Waiter–Client games are monotone in Waiter's bias q. In particular, essentially any Waiter–Client game $(X, \mathcal{F})$ has a threshold bias, that is, a unique positive integer $b_{\mathcal{F}}$ such that Client has a winning strategy for the $(1 : q)$ Waiter–Client game $(X, \mathcal{F})$ if and only if $q \ge b_{\mathcal{F}}$. From the way that Client–Waiter games have been defined, it is obvious that they also have a threshold bias (as observed in [4], this does not remain true if we require Waiter to offer exactly $p+q$ board elements per round). The threshold bias of the Client–Waiter game $(X, \mathcal{F})$ is the unique positive integer $b_{\mathcal{F}}$ such that Waiter has a winning strategy for the $(1 : q)$ game if and only if $q \ge b_{\mathcal{F}}$.

Waiter–Client and Client–Waiter games were first defined and studied by Beck under the names Picker–Chooser and Chooser–Picker, respectively (see, e.g., [2]). However, since *picking* and *choosing* are essentially the same, we feel that the names Waiter and Client, which first appeared in [5], help the reader to distinguish more easily between the roles of the two players.

As with Maker–Breaker and Avoider–Enforcer games, the probabilistic intuition turns out to be useful for Waiter–Client games as well. In particular, it is known to hold for the K_t-game (and in fact, for many other fixed graph games) [5], for the diameter two game (that is, the winning sets are the edge-sets of all subgraphs of K_n with diameter at most two) [14] and for the giant component game (that is, the game on $E(K_n)$ in which Waiter tries to force Client to build a connected component on as many vertices as possible) [6].

This paper is devoted to the study of several natural $(1 : q)$ Waiter–Client and Client–Waiter games, played on $E(K_n)$. For both the Waiter–Client and Client–Waiter versions, we will study the non-planarity game $(E(K_n), \mathcal{NP})$, where $\mathcal{NP}$ consists of all non-planar subgraphs of K_n, the K_t-minor game $(E(K_n), \mathcal{M}_t)$, where $\mathcal{M}_t$ consists of all subgraphs of K_n that admit a K_t-minor, and the non-k-colorability game $(E(K_n), \mathcal{NC}_k)$, where $\mathcal{NC}_k$ consists of all non-k-colorable subgraphs of K_n. The analogous Maker–Breaker and Avoider–Enforcer games were studied in [21].

It was proved in [8] that, if $q \ge n/2$, then when playing a $(1 : q)$ Maker–Breaker game on $E(K_n)$, Breaker can force Maker to build a forest, that is, Breaker has a winning strategy for the $(1 : q)$ game $\mathcal{M}_t$, for every $t \ge 3$. On the other hand, it was proved in [21] that, for every fixed $\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$ such that, if $q \le (1/2 - \varepsilon)n$, then Maker has a winning strategy for the $(1 : q)$ Maker–Breaker game $\mathcal{M}_t$ for every $t \le c\sqrt{n/\log n}$. The strict Avoider–Enforcer minor game was considered in [21] as well. It was proved there that, if $q \le (1/2 - \varepsilon)n$, then Enforcer has a winning strategy for