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Bounds on the maximum number of minimum dominating sets

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ABSTRACT

Given a graph with domination number γ, we find bounds on the maximum number of minimum dominating sets. First, for $\gamma \geq 3$, we obtain lower bounds on the number of γ-sets that do not dominate a graph on n vertices. Then, we show that γ-fold lexicographic product of the complete graph on $n^{1/\gamma}$ vertices has domination number γ and $\binom{n}{\gamma} - O(n^{\gamma-\frac{1}{\gamma}})$ dominating sets of size γ. Finally, we see that a certain random graph has, with high probability, (i) domination number γ; and (ii) all but $o(n^\gamma)$ of its γ-sets being dominating.

1. Introduction

A set S of vertices in a graph G is a dominating set if each vertex in the complement of S is adjacent to at least one vertex in S; the minimum cardinality $\gamma = \gamma(G)$ of such an S is called the domination number. A graph G with domination number γ thus has at least one dominating set of size γ and no dominating set of size at most $\gamma - 1$. It is easy to give an example of a graph with only one dominating set of size γ, but how *abundant* can the number of minimum dominating sets be? Let $X_\gamma = X_\gamma(G)$ be the number of dominating sets of size $\gamma(G)$. Then we let

$$M_{n,\gamma} = \max\{X_\gamma(G) : |V(G)| = n \text{ and } \gamma(G) = \gamma\}$$

be the maximum number of minimum dominating sets in a graph on n vertices and domination number γ. In view of the nature of the results in this paper, we will often state these in terms of $m_{n,\gamma}$, the minimum number of non-dominating γ-sets, since

$$m_{n,\gamma} = \min\left\{\binom{n}{\gamma} - X_\gamma(G) : |V(G)| = n \text{ and } \gamma(G) = \gamma\right\}.$$

Godbole et al. [5] provided a construction that gives a lower bound for $M_{n,\gamma}$; $\gamma \geq 3$. In this paper we use both probabilistic and constructive approaches to improve the results in [5] as well as to study the random variable $X_\gamma(G)$. Specifically,

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(i) We present results (Theorem 2.1 and Corollary 2.3) that show that for every $\gamma \geq 3$ and every G with dominating number γ, there are $\Omega(n^{\gamma-1-\frac{1}{\gamma-1}})$ non-dominating sets of size γ, and thus $m_{n,\gamma} \geq \Omega(n^{\gamma-1-\frac{1}{\gamma-1}})$;

(ii) In Proposition 2.2, we improve the constructions in [5] to produce a graph G with domination number γ for which the number of dominating sets is equal to $\binom{n}{\gamma} - O(n^{\gamma-\frac{1}{\gamma}})$, thus showing that $m_{n,\gamma} = O(n^{\gamma-\frac{1}{\gamma}})$;

(iii) Finally, in Theorem 3.1, we show that for every $\gamma \geq 3$, there exists an edge probability $p = p_n$ so that the random graph $G(n, p)$ satisfies, with high probability, the conditions $\gamma(G(n, p)) = \gamma$ and

$$X_\gamma(G(n,p)) = \binom{n}{\gamma} - O(n^{\gamma-\frac{1}{2\gamma-1}}).$$

The related algorithmic question of counting dominating sets using a "measure and conquer" approach together with linear programming is addressed in [3]. Counting dominating sets by taking advantage of the low treewidth of a graph is studied in [2].

Throughout the paper, γ will be a constant that does not depend on n. We will refer to a fraction that is asymptotically $(1-o(1))$ of all γ-sets as being "almost all" γ-sets. Moreover, it will be tacitly assumed in such situations that the domination number of the graph G in question is γ.

2. Lower bounds on the number of sets that do not dominate

We let $\mathcal{D}_G := A + I_n$, where A is the adjacency matrix corresponding to some ordering of the vertices of the graph. Thus a set of vertices $a_1, a_2, \ldots a_k$ in a graph G is a dominating set if and only if there are no rows in $\mathcal{D}_G$ with zeros in the a_ith column for all $1 \leq i \leq k$.

We start with the observation that for $a \leq n$, if we have $n\binom{a}{b} \geq \binom{n}{b}$, then it follows that

$$\begin{aligned} n\frac{a^b}{b!} &= \frac{a^b}{a(a-1)\ldots(a-b+1)} \cdot n \cdot \binom{a}{b} \\ &\geq \left(\prod_{i=1}^{b-1} \frac{1}{1-i/a}\right)\binom{n}{b} \\ &= \left(\prod_{i=1}^{b-1} \frac{1-i/n}{1-i/a}\right)\frac{n^b}{b!} \\ &\geq \frac{n^b}{b!}. \end{aligned}$$

Theorem 2.1. *For every $\gamma \geq 3$ and each graph G on n vertices with domination number γ, let $3 \leq k \leq \gamma + 1$. Then G must contain at least $\binom{n^{(k-2)/(k-1)}}{k}$ non-dominating sets of k vertices.*

Proof. Let r be the largest integer such that $\binom{n}{k-1} > n\binom{r}{k-1}$. This implies that $\binom{n}{k-1} \leq n\binom{r+1}{k-1}$, which, by the above observation (with $a = r + 1$ and $b = k - 1$) shows that

$$\frac{n(r+1)^{k-1}}{(k-1)!} \geq \frac{n^{k-1}}{(k-1)!},$$

and thus that $r + 1 \geq n^{(k-2)/(k-1)}$. We claim that there is a row in $\mathcal{D}_G$ with at least $r + 1$ zeros. Otherwise, each row induces at most $\binom{r}{k-1}$ non-dominating sets of size $k - 1$, and the total number of non-dominating sets of size $k - 1$ is at most $n\binom{r}{k-1} < \binom{n}{k-1}$, a contradiction to the fact that $\gamma > k - 1$. Using the row of $\mathcal{D}_G$ with $r + 1$ zeros, we can construct $\binom{r+1}{k} > \binom{n^{(k-2)/(k-1)}}{k}$ non-dominating sets of size k. This proves Theorem 2.1. $\square$

We next give an explicit construction of a graph G with $X_\gamma(G) = \binom{n}{\gamma} - O(n^{\gamma-\frac{1}{\gamma}})$. For n such that $n^{1/\gamma}$ is an integer, let $V(G) = \{1, 2, \ldots, n^{1/\gamma}\}^\gamma$. Let vertices $(u_1, \ldots, u_\gamma)$ and $(v_1, \ldots, v_\gamma)$ be adjacent if $u_i \neq v_i$ for each $i > 1$. G is the complete graph for $\gamma = 1$, and for $\gamma \geq 2$, G is the γ-fold lexicographic product of $K_{n^{1/\gamma}}$, in which the first coordinate is irrelevant as far as edges are concerned. For $\gamma = 2$, we consider the two-fold lexicographic product of $K_{\sqrt{n}}$, with $n = m^2$. A pair of vertices with the same second coordinate cannot dominate, because neither of the pair are adjacent to any other vertex with that second coordinate. Also, any pair of vertices with different second coordinate do dominate, since any other vertex has its second coordinate different from at least one of the second coordinates in the pair. This yields exactly $\sqrt{n}\binom{\sqrt{n}}{2} \sim n^{3/2}/2$ non-dominating pairs out of $\binom{n}{2} \sim n^2/2$ pairs. But our interest is in $\gamma \geq 3$, since we know from [5] that there is a graph