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Clique number of the square of a line graph

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ABSTRACT

We prove that the clique number of the square of a line graph of a graph G is at most $1.5\Delta_G^2$ and that the fractional strong chromatic index of G is at most $1.75\Delta_G^2$.

An *edge coloring* of a graph G is strong if each color class is an induced matching of G. The *strong chromatic index* of G, denoted by $\chi_s'(G)$, is the minimum number of colors for which G has a strong edge coloring. The strong chromatic index of G is equal to the chromatic number of the square of the line graph of G. The chromatic number of the square of the line graph of G is greater than or equal to the clique number of the square of the line graph of G, denoted by $\omega(L)$.

In this note we prove that $\omega(L) \leq 1.5\Delta_G^2$ for every graph G. Our result allows to calculate an upper bound on the fractional strong chromatic index of G, denoted by $\chi_{fs}'(G)$. We prove that $\chi_{fs}'(G) \leq 1.75\Delta_G^2$ for every graph G.

1. Introduction

A *strong edge coloring* of a graph G is an edge coloring in which every color class is an induced matching, that is, any two vertices that belong to distinct edges of the same color are not adjacent (in particular, no two edges of the same color intersect). The strong chromatic index of G, denoted by $\chi_s'(G)$, is the minimum number of colors in any strong edge coloring of G.

Concept of the strong edge coloring was introduced around 1985 by Erdős and Nešetřil [8]. They conjectured that for every graph G, with maximum degree Δ_G, $\chi_s'(G) \leq \frac{5}{4}\Delta_G^2$. The example of graph obtained from the cycle of length five by replacing each vertex by an independent set of size $\frac{\Delta}{2}$ shows that this bound, if true, is tight.

The trivial bound on the strong chromatic index of G is $2\Delta_G^2 - 2\Delta_G + 1$. Molloy and Reed [11] proved that $\chi_s'(G) \leq (2-\epsilon)\Delta_G^2$ for Δ_G sufficiently large, where ϵ is a small constant around $\frac{1}{50}$. Recently Bruhn and Joos [4] improved it to the $1.93\Delta_G^2$ for Δ_G sufficiently large.

We approach this problem from a different angle. A *line graph of* G is a graph whose each vertex represents an edge of G and two vertices are adjacent if and only if their corresponding edges are incident in G. A *square of a graph* H is a graph with the same set of vertices as H, in which two vertices are adjacent when their distance in H is at most 2. The strong chromatic index of the graph G is equal to the chromatic number of the square of the line graph of G (say L). The chromatic number of L is greater than or equal to the clique number of L (denoted by $\omega(L)$), so finding bounds on $\omega(L)$ is also an interesting problem. It is not known if the clique number of L is bounded by $\frac{5}{4}\Delta_G^2$ [1]. Chung et al. [6] proved that if L is a clique, then G has at most $\frac{5}{4}\Delta_G^2$ edges. Faudree et al. [9] proved that if G is a bipartite graph, then the clique number of L is at most Δ_G^2. The example of $K_{\Delta,\Delta}$ shows that this bound is tight. Recently Bruhn and Joos [4] proved that for G with $\Delta_G \geq 400$ the clique number of L is at most $1.74\Delta_G^2$. We improve this result.

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In this note, we focus on two topics related with the strong chromatic index of graphs. The first problem is finding the upper bound on the clique number of the square of the line graph of G. In our main theorem we prove an upper bound for the general case:

Theorem 1. *Let G be a simple graph and L be a square of the line graph of G. Then the clique number of L is at most $1.5\Delta_G^2$.*

We also present (Theorem 4) a new proof of known bounds for bipartite graphs (Δ_G^2). The second subject is a fractional strong chromatic index, denoted by $\chi'_{fs}(G)$. Using Theorem 1 we prove an upper bound on $\chi'_{fs}(G)$:

Theorem 2. *Let G be a simple graph. Then the fractional strong chromatic index of G is at most $1.75\Delta_G^2$.*

2. Upper bound on the clique number of the square of the line graph of *G*

For any two different edges from a graph G we define $dist_G(e, f)$ as the number of edges in the shortest path between e and f plus 1, i.e. $dist_G(e, f) = 1$ iff e and f intersect.

Remark 3. Let G be a simple graph, L be a square of the line graph of G and H be a subgraph of G, such that for each $e, f \in E(H)$ we have $dist_G(e, f) \le 2$. Then $\omega(L) = |E(H)|$.

2.1. Bipartite graphs

To better understand our method, first we present a new proof of a known bound for bipartite graphs. We use the same technique in the proof of our main theorem (which is more complicated).

Theorem 4. *Let G be a simple bipartite graph and L be a square of the line graph of G. Then the clique number of L is at most Δ_G^2.*

Proof. Let H be a subgraph of G, such that for each $e, f \in E(H)$ we have $dist_G(e, f) \le 2$. We show that H has at most Δ_G^2 edges.

Consider a vertex $v \in V(H)$ of degree Δ_H. Edges of the graph H can be divided into following sets (see Fig. 1):

1. $A = \{e \in E(H) : v \in e\}$, $|A| = \Delta_H$.
   A is the set of edges of H which are incident to v.
   Notice that all other edges of the graph H are in a distance at most 2 to all edges from A.
2. $B = \{e \in E(H) : e \notin A \wedge \exists_{f \in A} e \text{ and } f \text{ are adjacent}\}$, $|B| \le \Delta_H(\Delta_H - 1)$.
   B is the set of edges of H which are adjacent to edges from A and are not contained in A.
3. $C = \{e \in E(H) : \exists_{f \in E(G)} (v \in f \wedge f \notin E(H) \wedge e \text{ and } f \text{ are adjacent})\}$, $|C| \le (\Delta_G - \Delta_H)\Delta_H$.
   C is the set of edges of H which are adjacent to edges of $G - H$ which are incident to v.
4. $D = \{e \in E(H) : e \notin C \wedge \forall_{f \in A} dist_G(e, f) = 2\}$.
   D is the set of edges of H which are at the distance 2 to all edges from A and are not contained in C.
   Let S be the subgraph of H induced by D.
   We define a super vertex as a vertex which is adjacent in G to all neighbors of v from H, and it is not v. Because G is bipartite, each edge from S contains exactly one super vertex. We have at most $\Delta_G - 1$ super vertices in G. Furthermore each super vertex is incident in S to at most $\Delta_G - \Delta_H$ edges (because there are exactly Δ_H edges incident to this super vertex in G from neighbors of v from H). So the cardinality of D is at most

$$(\Delta_G - 1)(\Delta_G - \Delta_H).$$

Now we can sum up the number of edges in H

$$|E(H)| \le \Delta_H + \Delta_H(\Delta_H - 1) + (\Delta_G - \Delta_H)\Delta_H + (\Delta_G - 1)(\Delta_G - \Delta_H) = \Delta_G^2 - \Delta_G + \Delta_H \le \Delta_G^2. \quad \square$$

2.2. General case

In our proof we use the following lemma.

Lemma 5. *Let G be a graph with maximum degree Δ, p and w be integers such that $\Delta \le p \le w$ and $\Delta > w - p$. Consider p vertex covers of G (not necessarily different) such that each vertex cover contains at most w vertices. Moreover assume that for each vertex $v \in V(G)$ we have $deg(v) \le w - a$, where a is a number of vertex covers which contain v. Then the graph G has at most $w^2 - \frac{pw}{2}$ edges.*