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Note

Stability and Ramsey numbers for cycles and wheels

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ABSTRACT

We study the structure of red–blue edge colorings of complete graphs, with no copies of the n-cycle C_n in red, and no copies of the m-wheel $W_m = C_m * K_1$ in blue. Our first result is that, if we take $n = m$ and n odd, in any such coloring, one can delete at most two vertices to obtain a graph with a vertex-partition into three sets such that the edges inside the partition classes are red, and edges between partition classes are blue. As a second result, we obtain bounds for the Ramsey numbers $r(C_{2k+1}, W_{2j})$ for $k < j$, which asymptotically confirm the values of $4j + 1$, as conjectured by Zhang, Zhang and Chen.

1. Introduction

We study the structure of red–blue edge-colorings in complete graphs, which avoid certain monochromatic subgraphs. More concretely, we consider the case of odd positive integer n, and the forbidden monochromatic graphs given by the red n-cycle C_n and the blue n-wheel $W_n := C_n * K_1$. Our main result is the following:

Theorem 1. *Let $k \geq 6$ and $N \geq 5k + 3$. If $G := K_N$ has a red–blue coloring of its edges in a way such that C_{2k+1} is not a red subgraph of G and W_{2k+1} is not a blue subgraph of G, then there is a partition of $V(G)$ given by $\{U_0, U_1, U_2, U_3\}$ such that $|U_0| \leq 2$, $|U_i| \leq 2k$ for $1 \leq i \leq 3$; and every edge in $G - U_0$ inside the partition classes $\{U_1, U_2, U_3\}$ is red, and blue otherwise.*

A similar result was obtained by Nikiforov and Schelp [9], considering the case where the forbidden monochromatic subgraphs are odd cycles. More precisely, they proved that given $k \geq 2$ and $N \geq 3k + 2$, if a complete graph on N vertices has a red–blue coloring of its edges in a way such that C_{2k+1} is neither a red nor a blue subgraph of it, then there is a partition of its vertices given by $\{U_0, U_1, U_2\}$ such that $|U_0| \leq 1$ and the edges inside the partition classes U_1 and U_2 have one color; and are colored with the remaining color otherwise.

Our proof of Theorem 1 depends on certain bounds on asymmetric Ramsey numbers. In particular, it is known [4,5,11] that

$$r(C_n, W_m) = \begin{cases} 2n-1 & \text{for even } m, \text{ with } m \geq 4,\ n \geq 3m/2 - 1, \\ 3n-2 & \text{for odd } m, \text{ with } n \geq m \geq 3,\ (n, m) \neq (3, 3), \\ 2m+1 & \text{for odd } n, \text{ with } m \geq 3(n-1)/2,\ (n, m) \neq (3, 3), (n, m) \neq (3, 4), \\ 3n-2 & \text{for odd } n \text{ and } m; \text{ with } n < m \leq 3(n-1)/2. \end{cases}$$

Notice that $r(C_n, W_m)$ is not known for odd n and even m with $n < m < 3(n-1)/2$. Zhang, Zhang and Chen [11] raised a conjecture concerning these values.

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Conjecture 2 (*Zhang, Zhang and Chen [11]*). *Let n and m be integers such that $n < m$, with n odd and m even. Then $r(C_n, W_m) = 2m + 1$.*

From now on, suppose n and m are integers such that $n < m$, with $n = 2k + 1$ and $m = 2j$. We confirm the previous conjecture asymptotically in terms of j.

Theorem 3. *Let k and j be integers such that $2 < k < j$. We have the following bounds for the Ramsey number of the $(2k+1)$-cycle versus the $2j$-wheel.*

(a) *If $k \geq 3$, then $r(C_{2k+1}, W_{2j}) \leq \frac{9}{2}j + 1$.*
(b) *If $k \geq 3$, then $r(C_{2k+1}, W_{2j}) \leq 4j + 334$.*

In particular, we will make use of the upper bounds of $r(C_{2k+1}, W_{2k+2})$ for our proof of Theorem 1. Both bounds of Theorem 3 follow from a more general type of bound that we state and prove in Section 4.

2. Preliminaries

We fix a little bit of notation. For every graph G, we write $|G|$ and $\|G\|$ for its number of vertices and edges respectively. The *length* of a path P is $\|P\|$, its number of edges. For disjoint sets of vertices A and B, an *(A, B)-path* is a path with one endpoint in A, the other in B and no other vertices in $A \cup B$. For distinct vertices x and y, an *(x, y)-path* is an $(\{x\}, \{y\})$-path. For distinct sets of vertices A and B, a set of vertices S is a *(A, B)-separator* if every (A, B)-path intersects S. The well-known Menger's theorem (see e.g. [6, Chapter 3]) relates the size of a minimum (A, B)-separator with the maximum number of vertex-disjoint (A, B)-paths.

Theorem 4 (*Menger*). *Let A and B be subsets of vertices of a graph G. The size of any minimum (A, B)-separator in G equals the maximum number of vertex-disjoint (A, B)-paths in G.*

Given a red–blue coloring of the edges of a graph G, let G^R be the graph on $V(G)$ containing only the red-colored edges, similarly define G^B as the graph on $V(G)$ containing only the blue-colored edges. Let $E^R(G)$ and $E^B(G)$ be the set of edges of G^R and G^B, respectively.

Definition 5. Let G be a graph. A *semiclique* is a tuple (W, X), where $X \subseteq W \subseteq V(G)$, X induces a complete subgraph and the edges in $E(W \setminus X, X)$ induce a complete bipartite subgraph.

The notion of semicliques will be useful in the proof of Theorem 1. The main property of semicliques is that every pair of vertices can be joined by paths of various lengths, and that allows us to find cycles of various sizes.

Lemma 6. *Let (W, X) a semiclique. Then every pair of distinct vertices in W can be joined by paths of every length between 2 and $|X| - 1$.*

Proof. As X induces a complete subgraph, every pair of distinct vertices in X can be joined by paths of every length between 1 and $|X| - 1$. For distinct pair of vertices in W, not necessarily contained in X, we can use the edges in $E(W \setminus X, X)$ to extend the mentioned paths or to find a pair of length 2 connecting these vertices, and conclude the result. □

Corollary 7. *For every $i \in \{1, 2\}$, let (R_i, S_i) be semicliques in a graph G, such that $R_1 \cap R_2 = \emptyset$. Suppose that $\min\{|S_1|, |S_2|\} \geq 3$ and that there exist two disjoint edges in $E(R_1, R_2)$. Then G contains cycles of every length between 6 and $|S_1| + |S_2|$.*

Proof. Using Lemma 6 we can join every two vertices in a semiclique with paths of various lengths. Choosing these vertices to be the endpoints of two disjoint edges in $E(R_1, R_2)$ we find cycles of the desired lengths. □

We shall make use of the values of Ramsey numbers for cycles, which are completely known.

Theorem 8 (*Faudree and Schelp [8]*). *We have*

$$r(C_n, C_m) = \begin{cases} 6 & (n, m) \in \{(3, 3), (4, 4)\}, \\ 2n - 1 & 3 \leq m \leq n, \text{ odd } m, (n, m) \neq (3, 3), \\ n + \dfrac{m}{2} - 1 & 4 \leq m < n \text{ both } n, m \text{ even}, (n, m) \neq (4, 4), \\ \max\left\{n + \dfrac{m}{2} - 1, 2m - 1\right\} & 3 \leq m \leq n, \text{ even } m \text{ and odd } n. \end{cases}$$

Theorem 9 (*Surahmat, Baskoro and Tomescu [10]*). *We have that $r(C_{2k+1}, W_{2k+1}) = 6k + 1$, for all integers $k \geq 1$.*

Next, we need some results on the stability of cycle-forbidding red–blue colorings, as shown by Nikiforov and Schelp [9].