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Characterization of graphs with given order, given size and given matching number that minimize nullity

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ABSTRACT

Let $G = (V(G), E(G))$ be a finite undirected graph without loops and multiple edges, $c(G) = |E(G)| - |V(G)| + \theta(G)$ be the dimension of cycle spaces of G with $\theta(G)$ the number of connected components of G, $m(G)$ be the matching number of G, and $\eta(G)$ be the nullity of G. It was shown in Wang and Wong (2014) that the nullity $\eta(G)$ of G is bounded by an upper bound and a lower bound as

$$|V(G)| - 2m(G) - c(G) \leq \eta(G) \leq |V(G)| - 2m(G) + 2c(G).$$

Graphs with nullity attaining the upper bound have been characterized by Song et al. (2015). However, the problem of characterization of graphs whose nullity attain the lower bound is left open till now.

In this paper, we are devoted to solve this open problem, proving that $\eta(G) = |V(G)| - 2m(G) - c(G)$ if and only if the cycles (if any) of G are pairwise vertex-disjoint and G can be switched, by a series of operations of deleting a pendant vertex together with its adjacent vertex, to an induced subgraph of G, which is the disjoint union of $c(G)$ odd cycles and $|V(G)| - 2m(G) - c(G)$ isolated vertices.

1. Introduction

Let G be a finite undirected graph without loops and multiple edges. The adjacency matrix $A(G)$ of G with vertex set $V(G) = \{v_i : 1 \leq i \leq n\}$ and edge set $E(G)$ is defined to be an $n \times n$ symmetric matrix $[a_{ij}]$ such that $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The rank $r(G)$ of G is defined to be the rank of $A(G)$ and the nullity $\eta(G)$ of G is defined to be the multiplicity of 0 as an eigenvalue of $A(G)$. Obviously,

$$r(G) + \eta(G) = |V(G)|.$$

The dimension of cycle spaces of G, written as $c(G)$, is defined as

$$c(G) = |E(G)| - |V(G)| + \theta(G),$$

where $\theta(G)$ denotes the number of connected components of G. When G is connected, then G is a tree if $c(G) = 0$; G is a unicyclic graph if $c(G) = 1$; and G is a bicyclic graph if $c(G) = 2$.

A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Further, H is called an induced subgraph of G if two vertices of $V(H)$ are adjacent in H if and only if they are adjacent in G. The number of neighbors of a vertex x in G is called the degree of x, which is written as $d(x)$. If $d(x) = 1$, we call x a pendant vertex of G. For a subset U of $V(G)$, denote by $G - U$

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the graph obtained from G by removing the vertices of U together with all edges incident to them. Sometimes we use the notation $G - G_1$ instead of $G - V(G_1)$ if G_1 is an induced subgraph of G. A vertex $x \in V(G)$ (resp., An edge $e \in E(G)$) is called a cut-point (resp., cut-edge) of G if the resultant graph $G - x$ (resp., $G - e$) has more components than those of G. For an induced subgraph G_1 and a vertex x outside G_1, the induced subgraph of G with vertex set $V(G_1) \cup \{x\}$ is simply written as $G_1 + x$. A matching M in G is a set of pairwise non-adjacent edges, that is, no two edges share a common vertex. A maximum matching is a matching that contains the largest possible number of edges. The matching number of G, denoted by $m(G)$, is the size of a maximum matching. By C_n (resp., P_n, K_n) we denote a cycle (resp., a path, a complete graph) on n vertices.

A subgraph H of G is called an elementary subgraph if each component of H is a single edge or a cycle. We use $p(H)$ (resp., $s(H)$) to denote the number of components (resp. the number of cycles) in an elementary subgraph H. The characteristic polynomial of G, denoted by $\varphi(G, x)$, is defined as $\det(xI - A(G))$, where I is an identity matrix of order the same as that of $A(G)$.

The chemical importance of the nullity of graphs lies in the fact, that within the Höckel molecular orbital model, if the nullity $\eta(G)$ of a molecular graph G is positive, then the corresponding chemical compound is highly reactive and unstable, or nonexistent (see [1,7]). In 1957, Collatz and Sinogowitz [6] posed the problem of characterizing all singular graphs. The problem is very hard and only some particular results are known (see [3–5,8–22]). In studying above problem, some authors focused on describing the nullity set of certain class of graphs in terms of the order of graphs. Let $\mathcal{G}_n$ be a set of a class of graphs of order n, and let $[0, n] = \{0, 1, 2, \ldots, n\}$. A subset N of $[0, n]$ is said to be the nullity set of $\mathcal{G}_n$ if $\eta(G) \in N$ for each $G \in \mathcal{G}_n$, and for each $k \in N$ there exists at least one graph $G \in \mathcal{G}_n$ such that $\eta(G) = k$. It is well known that the nullity of a tree T of order n is $n - 2m(T)$ (see [7]). The nullity set of unicyclic graphs of order $n \geq 5$ was proved in [20] to be $[0, n - 4]$. It was shown in [13] that the nullity set of bicyclic graphs of order n is $[0, n - 2]$. It was shown in [5] that the nullity set of tricyclic graphs of order $n \geq 8$ is $[0, n - 4]$. For bipartite graphs of order n it was shown in [8,18] that the nullity set is $\{n - 2k : k = 0, 1, 2, \ldots, \lfloor n/2 \rfloor\}$.

Recently, Wang et al. [21] obtained the following bounds for the matching number of G:

$$\left\lceil \frac{r(G) - c(G)}{2} \right\rceil \leq m(G) \leq \left\lfloor \frac{r(G) + 2c(G)}{2} \right\rfloor.$$

They also showed that their bounds are best possible by providing examples. The bounds for the matching number can be rewritten in an equivalent form as bounds for the nullity of G:

Proposition 1.1. *[Theorem 1.1, [21]] For any graph G,*

$$|V(G)| - 2m(G) - c(G) \leq \eta(G) \leq |V(G)| - 2m(G) + 2c(G).$$

Graphs with nullity attaining the upper bound in Proposition 1.1 have been characterized by Song et al. in [19]. In order to state the main result in [19], we need to introduce an acyclic graph T_G (see [19] or [21]) for a graph G whose cycles are pairwise vertex-disjoint. Let G be a graph with pairwise vertex-disjoint cycles, and let $\mathcal{C}(G)$ denote the set of all cycles in G. The acyclic graph T_G is obtained from G by contracting cycles. More specifically, the vertex set $V(T_G)$ is taken to be $U \cup W_G$, where U consists of all vertices of G that do not lie on a cycle and $W_G = \{v_C : C \in \mathcal{C}(G)\}$. Two vertices in U are adjacent in T_G if and only if they are adjacent in G, a vertex $u \in U$ is adjacent to a vertex $v_C \in W_G$ if and only if u is adjacent (in G) to a vertex in the cycle C, and vertices v_C, v_O are adjacent in T_G if and only if there exists an edge in G joining a vertex of $C \in \mathcal{C}(G)$ to a vertex of $O \in \mathcal{C}(G)$. It is clear that T_G is always acyclic and is a tree when G is connected. Since the cycles of G are pairwise vertex-disjoint, deleting edges from G, one from each cycle, will result in a forest F and as $c(F) = 0$ it follows that G has altogether $c(G)$ cycles. Note that the graph $T_G - W_G$ (obtained from T_G by deleting vertices in W_G and the incident edges) is the same as the graph obtained from G by deleting all cycles and the incident edges.

Proposition 1.2 (*Theorem 1.4, [19]*). *For any graph G, $\eta(G) = |V(G)| - 2m(G) + 2c(G)$ if and only if the following three conditions are all satisfied:*

(a) *Distinct cycles of G (if any) are vertex-disjoint;*
(b) *The length of each cycle of G (if any) is a multiple of 4;*
(c) *$m(T_G) = m(T_G - W_G)$ or, equivalently, there exists a maximum matching of T_G that does not cover any vertex in W_G.*

However, the problem of characterization of graphs whose nullity attain the lower bound are left open till now. The purpose of this work is to characterize graphs G that satisfy $\eta(G) = |V(G)| - 2m(G) - c(G)$. For convenience, hereafter in this paper we say *a graph G satisfies the minimal nullity condition* if $\eta(G) = |V(G)| - 2m(G) - c(G)$.

In view of the following known result, every acyclic graph satisfies the minimal nullity condition:

Proposition 1.3 (*[7]*). *For every acyclic graph T with at least one vertex, $\eta(T) = |V(T)| - 2m(T)$.*

Unicyclic graphs that satisfy the minimal nullity condition are also characterized (due to Guo, Yan and Yeh). The following result is implicit in [11]: