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Note

Spanning trees with bounded degrees and leaves

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ABSTRACT

Rivera-Campo provided a degree sum condition for a graph to have a spanning tree with bounded degrees and leaves. In this paper, we give an independence number condition for a graph to have a spanning tree with bounded degrees and leaves, which also partially solves the conjecture made by Enomoto and Ozeki (2010).

1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the *order* of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v and by $N_G(v)$ the neighborhood of v. Thus $\deg_G(v) = |N_G(v)|$. An edge joining two vertices u and v is denoted by uv or vu. The *independence number* and the *connectivity* of G are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. Let T be a tree. A vertex of T with degree one is often called a *leaf*, and the set of leaves of T is denoted by *Leaf* (T). For a set X, the cardinality of X is denoted by $|X|$ or $\#X$.

Chvátal and Erdös [2] showed that if $\alpha(G) \le \kappa(G) + 1$, then G has a hamiltonian path. This result was generalized to a spanning k-tree as follows, where a *k-tree* is a tree with maximum degree at most k.

Theorem 1 (*Neumann-Lara and Rivera-Campo [6]*). *Let $k \ge 2$ be an integer and G be a connected graph. If $\alpha(G) \le (k-1)\kappa(G) + 1$, then G has a spanning k-tree.*

On the other hand, a hamiltonian path is a spanning tree having exactly two leaves. From this point of view, Win provided a sufficient condition for a graph to have a spanning tree having a small number of leaves.

Theorem 2 (*Win [9]*). *Let $k \ge 2$ be an integer and G be a connected graph. If $\alpha(G) \le \kappa(G) + k - 1$, then G has a spanning tree having at most k leaves.*

Recently, Rivera-Campo obtained a degree sum condition for a graph to have a spanning tree with bounded degree as well as with a small number of leaves.

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Theorem 3 (*Rivera-Campo [8]*). *Let p, n and $d_1, d_2, \ldots, d_p$ be integers such that $1 \le n \le p-1$ and $2 \le d_1 \le d_2 \le \cdots \le d_p \le p-1$. Let G be an n-connected graph of order p with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. If*

$$\deg_G(x) + \deg_G(y) \ge p - 1 - \sum_{j=1}^{n}(d_j - 2)$$

for any non-adjacent vertices x and y of G, then G has a spanning tree T that has at most $\sum_{j=1}^{n}(d_j-2)+2$ leaves and satisfies $\deg_T(v_i) \le d_i$ for all $i = 1, 2, \ldots, p$.

For a function $f : V(G) \to \{1, 2, 3, \ldots\}$, a spanning tree T of a graph G is called a *spanning f-tree* if $\deg_T(v) \le f(v)$ for all vertices v of G. Here, we give a sufficient condition using independence number for a graph to have a spanning f-tree with a small number of leaves. The following is our result.

Theorem 4. *Let $n \ge 1$ be an integer. Let G be an n-connected graph and $f : V(G) \to \{2, 3, 4, \ldots\}$ be a function. If*

$$\alpha(G) \le \min_X \left\{ \sum_{x \in X}(f(x) - 1) : X \subseteq V(G) \text{ and } |X| = n \right\} + 1, \tag{1}$$

then G has a spanning f-tree that has at most $\min_X\{\sum_{x\in X}(f(x)-2) : X \subseteq V(G)$ and $|X| = n\} + 2$ leaves.

We now give some other results and a conjecture related to our theorem, and explain the relation between our theorem and them. The following theorem gives a sufficient condition for a graph to have a spanning tree that contains specified vertices as leaves.

Theorem 5 (*Matsuda and Matsumura [5]*). *Let n, k and s be integers such that $k \ge 2$, $0 \le s \le k$ and $s \le n-1$, and let G be an n-connected graph. If $\alpha(G) \le (n-s)(k-1)+1$, then for any s vertices of G, G has a spanning k-tree that includes the s specified vertices as leaves.*

Enomoto and Ozeki made the following conjecture on a spanning f-tree from the above theorem, and partially solved it (Theorem 7).

Conjecture 6 (*Enomoto and Ozeki [3]*). *Let $n \ge 1$ be an integer, G be an n-connected graph and $f : V(G) \to \{1, 2, 3, \ldots\}$ be a function. If $\sum_{x\in V(G)} f(x) \ge 2(|G|-1)$ and*

$$\alpha(G) \le \min_X \left\{ \sum_{x \in X}(f(x) - 1) : X \subseteq V(G) \text{ and } |X| = n \right\} + 1,$$

then G has a spanning f-tree.

Theorem 7 (*Enomoto and Ozeki [3]*). *Let $n \ge 1$ be an integer, G be an n-connected graph and $f : V(G) \to \{1, 2, 3, \ldots\}$ be a function. If $\#\{v \in V(G) : f(v) = 1 \text{ or } 2\} \le n+1$, $\sum_{x\in V(G)} f(x) \ge 2(|G|-1)$ and*

$$\alpha(G) \le \min_X \left\{ \sum_{x \in X}(f(x) - 1) : X \subseteq V(G) \text{ and } |X| = n \right\} + 1,$$

then G has a spanning f-tree.

Note that Conjecture 6 is a generalization of Theorem 5 since by setting $f(u) = 1$ for s specified vertices u and $f(v) = k$ for the other vertices v, Conjecture 6 implies Theorem 5. Moreover, another conjecture related to Conjecture 6 was proposed by Ozeki and Yamashita [7] using a new notation $Cut(G, f)$ instead of the connectivity. Our Theorem 4 solves the conjecture under the assumption that $f(v) \ge 2$ for all vertices v, and the proof techniques in this paper are different from those in Enomoto and Ozeki [3].

Some other results on spanning trees related to our theorem are given in [1], and many current results on spanning trees can be found in [7].

2. Proof of Theorem 4

In order to prove Theorem 4, we need the following Lemmas.

Lemma 1 (*Kouider [4]*). *Let H be a subgraph of an n-connected graph G, where $n \ge 2$. Then either $V(H)$ is covered by a cycle of G, or there is a cycle C in G such that $\alpha(H - V(C)) \le \alpha(H) - n$. In particular, if $\alpha(H) \le n$, then $V(H)$ is covered by a cycle of G.*

By Lemma 1, we can easily obtained following lemma.