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Total-colorings of complete multipartite graphs using amalgamations

Aseem Dalal [a], B.S. Panda [a], C.A. Rodger [b,*]

[a] Department of Mathematics, Indian Institute of Technology Delhi, Delhi-110016, India

[b] Department of Mathematics and Statistics, Auburn University, AL, 36849-5310, USA

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ABSTRACT

This paper makes progress towards settling the long-standing conjecture that the total chromatic number χ'' of the complete p-partite graph $K = K(r_1, \dots, r_p)$ is $\Delta(K) + 1$ if and only if $K \neq K_{r,r}$ and if K has an even number of vertices then $\Sigma_{v \in V(K)}(\Delta(K) - d_K(v))$ is at least the number of parts of odd size. Graphs of even order that are fairly close to being regular are the ones for which $\chi''(K)$ remains in doubt. In this paper we show that K is of Type 1 if $|V(K)|$ is even and $r_2 < r_3$ (with parts arranged in non-decreasing order of size), thereby improving on the result of Dong and Yap published in 2000. Furthermore, it is shown using this result together with the novel approach of graph amalgamations that all complete multipartite graphs of the form $K(r, r, \dots, r, r+1)$ are of Type 1.

1. Introduction and definitions

A graph $G = (V, E)$ is said to have a total coloring if the elements of $V(G) \cup E(G)$ are colored so that: adjacent vertices receive different colors; incident edges receive different colors; and if edge e is incident with vertex v then e and v receive different colors. The total chromatic number $\chi''(G)$ is the least number of colors needed to totally color G. The complete p-partite graph $K = K[V_1, \dots, V_p]$ is the simple graph with vertex set $V(K) = \cup_{i=1}^{p} V_i$ (each set V_i is called a part) in which two vertices are joined if and only if they occur in different parts of K. If the names of the vertex sets are unimportant then K is simply referred to as $K(r_1, \dots, r_p)$, where $|V_i| = r_i$ for $1 \leq i \leq p$.

The graph K is of sufficient complexity that settling the values of its graph parameters is often a challenge. Finding the chromatic index $\chi'(K)$ is a typical example. Of course the classic result of Vizing shows that $\chi'(G)$ is $\Delta(G)$ or $\Delta(G) + 1$, thereby giving rise to the classification of whether a graph is Class 1 or Class 2 respectively. It was finally shown in 1992 that K is a Class 2 graph if and only if it is overfull [10]. Similar to Vizing's result, it is conjectured that the value of $\chi''(G)$ for any simple graph G is either $\Delta(G) + 1$ or $\Delta(G) + 2$ (see [1,14]), and G is said to be of Type 1 or Type 2 respectively based on this value. Bermond settled the type of K when it is regular [2]. Yap [17] proved that $\chi''(K) \leq \Delta + 2$, and with Chew [6] showed that if K has an odd number of vertices or if $p = 3$ then it is Type 1. Along the lines of the result proved in this paper, in 1992 Chew and Yap [6] also showed the following result.

Theorem 1.1. *If either $r_1 < r_2 \leq r_3 \leq \cdots \leq r_p$ or $p = 3$, then $K(r_1, \dots, r_p)$ is of Type 1.*

It was not until 2000 that Dong and Yap [8] improved this result, showing the following.

* Corresponding author.
E-mail addresses: aseem.dalal@gmail.com (A. Dalal), bspanda@maths.iitd.ac.in (B.S. Panda), rodgec1@auburn.edu (C.A. Rodger).

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Theorem 1.2 ([8]). *Suppose that $r_1 \leq r_2 \leq \cdots \leq r_p$ and that $|V(K)| = 2n$. If $r_2 \leq r_3 - 2$ then K is of Type 1. Also if K is not regular with $p = 4$ then it is of Type 1.*

The proof techniques in all these papers are very similar, though get more complicated as more difficult cases are attacked. They build upon the idea of coloring the vertices so that all vertices in one part, say V_β, receive the same color, while all other vertices receive different colors (a so-called β-biased total coloring). Such total colorings were characterized in [11] by Hoffman and Rodger, thereby producing the following theorem. It is most easily stated in terms of the deficiency, which is the measure of how far a graph G is from being regular, and is defined by:

$$def(G) = \Sigma_{v \in V(G)}(\Delta(G) - d_G(v)).$$

Theorem 1.3. *Suppose that $r_1 \leq r_2 \leq \cdots \leq r_p$ and that $|V(K)| = 2n$. If*

$$def(K) \geq \begin{cases} 2n - p_1 & \text{if } p = 2 \text{ or} \\ & \text{if } p \text{ is even, } r_1 \text{ is odd, and } r_1 = r_{p-1}, \\ 2n - p_r & \text{otherwise,} \end{cases}$$

then K is Type 1.

Knowing more about edge-coloring results certainly helps with attacking total chromatic number problems. Indeed, the proof of Theorem 1.3 made use of the following result, also by Hoffman and Rodger [9]. The subgraph induced by the vertices of maximum degree in a graph G is known as its core, G_Δ.

Theorem 1.4 ([9]). *If G is a simple graph in which G_Δ is a forest then G is a Class 1 graph.*

Along the lines of Theorem 1.4, Xie and Yang [15] proved the following theorem for graphs with even order and high degree.

Theorem 1.5 ([15]). *Let $G \neq K_2$ be a graph of even order and G_Δ be a forest. If $\delta(G)+\Delta(G) \geq \frac{3}{2}|V(G)|-\frac{3}{2}$, then $\chi''(G) = \Delta+1$.*

Recently, Dalal and Rodger [7] introduced a novel approach using amalgamations to attack the problem of finding the total chromatic number of the complete multipartite graph. They exemplified the power of the approach by settling the classification problem for all complete 5-partite graphs, thereby extending the result for $p = 4$ in Theorem 1.2. More precisely, they proved:

Theorem 1.6 ([7]). *The graph $K = K(r_1, \ldots, r_5)$ is Type 2 if and only if $|V(K)| \equiv 0 \pmod 2$ and $def(K)$ is less than the number of parts in K of odd size.*

The idea of the technique is to reverse the process of taking a graph homomorphism. Suppose that we have a total coloring of K in which all vertices in V_i receive the same color; then we apply a homomorphism that maps each vertex in V_i to a single new vertex v_i. This totally colored multigraph K', the so-called natural amalgamation of K, is much easier to total color than it is to color K. The following result shows that if we can just total color the homomorphic image, then it is possible to disentangle the vertices to produce K.

Theorem 1.7 ([12]). *Let $1 \leq r_1 \leq r_2 \leq \cdots \leq r_p$ and let G be the multigraph on the p vertices $v_1, \ldots, v_p$ in which v_i is joined to v_j with r_ir_j edges. Suppose there exists a k-edge-coloring of G in which each vertex v_i is incident with $x_{i,j} \leq r_i$ edges of color j. Then there exists a proper k-edge-coloring of $K(r_1, \ldots, r_p)$ in which for $1 \leq i \leq p$ the number of vertices in V_i incident with edges colored j is exactly $x_{i,j}$.*

Note that in this result, each color class in $K(r_1, \ldots, r_p)$ is a matching. Bryant et al. [4] gave the following necessary and sufficient condition for decomposition of complete multigraphs into cycles of varying lengths, which will also be of use in our proofs. For any graph G, let $r * G$ denote the multigraph formed by replacing each edge in G with r edges.

Theorem 1.8 ([4]). *Let λ, n and m be integers with $n, m \geq 3$ and $\lambda \geq 1$. There exists a decomposition of $\lambda * K_n$ into m-cycles if and only if*

(1) $m \leq n$;
(2) $\lambda(n-1)$ *is even; and*
(3) m *divides* $\lambda\binom{n}{2}$.

There exists a decomposition of $\lambda * K_n$ into m-cycles and a perfect matching if and only if

(1) $m \leq n$;
(2) $\lambda(n-1)$ *is odd; and*
(3) m *divides* $\lambda\binom{n}{2} - \frac{n}{2}$.