Discrete Mathematics 339 (2016) 1600–1604

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Element deletion changes in dynamic coloring of graphs

Lian-Ying Miao [a], Hong-Jian Lai [b,*], Yan-Fang Guo [a], Zhengke Miao [c]

[a] *Institute of Mathematics, China University of Mining and Technology, Xuzhou 221116, China*
[b] *Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA*
[c] *School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu, 221116, China*

ARTICLE INFO

Article history:
Received 9 July 2015
Received in revised form 12 January 2016
Accepted 12 January 2016
Available online 4 February 2016

Keywords:
Dynamic coloring
Dynamic chromatic number

ABSTRACT

A proper vertex k-coloring of a graph G is dynamic if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic k-coloring is the dynamic chromatic number $\chi_d(G)$. In this paper the differences between $\chi_d(G)$ and $\chi_d(G-e)$, and between $\chi_d(G)$ and $\chi_d(G-v)$ are investigated respectively.

1. Introduction

In this paper, all graphs $G = (V, E)$ are finite, simple and undirected. For $v \in V$, $N_G(v)$ is the set of vertices adjacent to v, and the degree of v, denoted by $d_G(v)$, is $|N_G(v)|$. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum degree and minimum degree of G, respectively. When the graph G is understood from the context, we often omit the subscript G, and use δ, Δ for $\delta(G)$, $\Delta(G)$, respectively. If $uv \in E$, then u is a **neighbor** of v. For $W \subseteq V$, $G - W$ denotes the graph obtained from G by deleting the vertices in W together with their incident edges. If $W = \{w\}$, we often write $G - w$ for $G - \{w\}$. If $U \subseteq V$, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U. Let C_n and P_n denote a cycle and a path on n vertices, respectively. In a graph G, an **elementary subdivision** of an edge $e = uv \in E(G)$ is the operation of replacing e with a path uv_ev through a new vertex v_e. A graph H is a **subdivision** of a graph G if H can be obtained from G by a sequence of elementary subdivisions. For a real number x, we use $\lceil x \rceil$ to denote the least integer no less than x.

For an integer $k > 0$, let $\overline{k} = \{1, 2, \ldots, k\}$. If $S \subseteq V(G)$ is a subset and $c : V(G) \mapsto \overline{k}$ is a mapping, then define $c(S) = \{c(x) : x \in S\}$. A **dynamic k-coloring** of a graph G is a mapping $c : V(G) \mapsto \overline{k}$ satisfying both of the following:

(C1) If $uv \in E(G)$, then $\varphi(u) \neq \varphi(v)$, and

(C2) for each vertex $v \in V(G)$, $|c(N(v))| \geq \min\{2, d_G(v)\}$.

The **dynamic chromatic number** $\chi_d(G)$ is the smallest integer k such that G has a dynamic k-coloring. Dynamic coloring was first introduced in [12,9], and is a special case of the r-hued colorings [8,7,13] when $r = 2$. The study of dynamic coloring has drawn lots of attention, as seen in [1–6,8,9,12,10,11,13,14], among others.

Unlike classic colorings, a subgraph of a graph G may have a bigger dynamic chromatic number than G. A natural problem is to investigate the differences between $\chi_d(G)$ and $\chi_d(G-e)$, and between $\chi_d(G)$ and $\chi_d(G-v)$. This motivates the current study. In Section 2, we will investigate the best possible bounds for the differences between $\chi_d(G-e)$ and $\chi_d(G)$, and between $\chi_d(G-v)$ and $\chi_d(G)$.

* Corresponding author.
E-mail addresses: miaolianying@cumt.edu.cn (L.-Y. Miao), hongjianlai@gmail.com (H.-J. Lai), 243834696@qq.com (Y.-F. Guo), zkmiao@jsnu.edu.cn (Z. Miao).

http://dx.doi.org/10.1016/j.disc.2016.01.009
0012-365X/

2. Comparisons between $\chi_d(G)$ and $\chi_d(G-e)$, and between $\chi_d(G)$ and $\chi_d(G-v)$

It is well known that if H is a subgraph of a graph G, then $\chi(G) \geq \chi(H)$. However, there exist graphs G with a subgraph H such that $\chi_d(H) > \chi_d(G)$. For example, let G be the 5-cycle with one chord, and let H be the 5-cycle, then it is routine to verify that $\chi_d(G) = 4$ but $\chi_d(H) = 5$.

In this section, we investigate tight bounds for the change of the dynamic chromatic number when an edge or a vertex is being removed. We start with a lemma, which follows from definition immediately.

Lemma 2.1. *If G is a connected graph on at least 2 vertices, then $\chi_d(G) \leq 2$ is and only if $G \in \{K_1, K_2\}$.*

Theorem 2.1. *Each of the following holds.*

(i) *Let G be a connected graph with $|V(G)| \geq 3$. Then for any edge $e = uv \in E(G)$,*

$$\chi_d(G) - 2 \leq \chi_d(G-e) \leq \chi_d(G) + 2. \tag{1}$$

(ii) *There exists a graph G such that $\chi_d(G-e) = \chi_d(G) + 2$ for at least one edge $e \in E(G)$.*

(iii) *If a connected graph G satisfies that $\chi_d(G-e) = \chi_d(G) - 2$ for at least one edge e in G, then $G = C_5$.*

Proof. (i) Let $k_1 = \chi_d(G-e)$, and let $c_1 : V(G-e) \mapsto \overline{k}_1$ be a dynamic k_1-coloring of $G-e$. Obtain a new coloring c_1' from c_1 by defining

$$c_1'(z) = \begin{cases} c_1(z) & \text{if } z \notin \{u, v\} \\ k_1 + 1 & \text{if } z = u \\ k_1 + 2 & \text{if } z = v. \end{cases}$$

By definition, $c_1' : V(G) \mapsto \overline{k_1 + 2}$ is a dynamic $(k_1 + 2)$-coloring of G, and so $\chi_d(G) \leq \chi_d(G-e) + 2$.

Now let $k_2 = \chi_d(G)$ and $c_2 : V(G) \mapsto \overline{k}_2$ be a dynamic k_2-coloring of G. Since $|V(G)| \geq 3$ and since G is connected, there exists $x \in N_G(u) - \{v\}$ or $y \in N_G(v) - \{u\}$. Choose such x and y so that $|\{x, y\}|$ is maximized. If $|\{x, y\}| = 1$, then by the maximality of $|\{x, y\}|$, and since G is connected, we must have $d_G(u) \leq 2$ and $d_G(v) \leq 2$. In this case, we have $\chi_d(G) = \chi_d(G-e)$, and so $\chi_d(G) \leq \chi_d(G-e)+2$. Hence we assume that $x \neq y$. Obtain a new coloring c_2' from c_2 by defining

$$c_2'(z) = \begin{cases} c_2(z) & \text{if } z \notin \{x, y\} \\ k_2 + 1 & \text{if } z = x \\ k_2 + 2 & \text{if } z = y. \end{cases}$$

By definition, $c_2' : V(G-e) \mapsto \overline{k_2 + 2}$ is a dynamic $(k_2 + 2)$-coloring of $G - e$, and so $\chi_d(G-e) \leq \chi_d(G)+2$. This proves (i).

(ii) For an integer $r \geq 4$, let H be a complete r-partite graph with partite sets $V_1, V_2, \ldots, V_r$, such that $|V_i| \geq 2$ for each i with $1 \leq i \leq r$, and let u and v be two new vertices. Let G be the graph obtained from H by adding a new edge uv to H and by joining u to every vertex in V_1 and joining v to every vertex in V_2. It is routine to verify that $\chi_d(G) = \chi(G) = r$, and that $\chi_d(G-uv) = r+2$, since the vertices in each of V_1 and V_2 must be colored with at least two colors.

(iii) Let G be a connected graph with at least one edge such that $\chi_d(G-e) = \chi_d(G) - 2$ for some edge $e = uv \in E(G)$, and let $k = \chi_d(G-e)$. If $\chi_d(G-e) \leq 2$, then by Lemma 2.1, $G \in \{K_2, P_3\}$, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)-2$ for some $e \in E(G)$. Hence we assume that $k = \chi_d(G-e) \geq 3$.

Let $c : V(G-e) \mapsto \overline{k}$ be a dynamic k-coloring. Assume without loss of generality, that $d_G(u) \geq d_G(v)$. If $d_G(v) = 1$, then v is an isolated vertex of $G - e$. As $k \geq 3$, we can pick a vertex $u' \in N_G(u) - \{v\}$ and redefine $c(v) \in \overline{k} - \{c(u), c(u')\}$ to obtain a k-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. If $d_G(u) \geq 3$, then by $k \geq 3$, we can redefine $c(u) = k+1$ to obtain a $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G) - 2$. Hence we may assume that $d_G(u) = d_G(v) = 2$. Let $N_G(u) = \{v, u'\}$, $N_G(v) = \{u, v'\}$. We have the following claims.

Claim 1. $u' \neq v'$.

If $u' = v'$, then obtain a new coloring c' from c by defining

$$c'(z) = \begin{cases} c(z) & \text{if } z \neq u \\ k+1 & \text{if } z = u. \end{cases}$$

By definition, $c' : V(G) \mapsto \overline{k+1}$ is a dynamic $(k+1)$-coloring of G, contrary to the assumption that $\chi_d(G-e) = \chi_d(G)-2$. Thus Claim 1 must hold.

Claim 2. $c(u) = c(v') \neq c(u') = c(v)$.

If $c(u) \neq c(v')$, then obtain a new coloring c'' from c by defining

$$c''(z) = \begin{cases} c(z) & \text{if } z \neq v \\ k+1 & \text{if } z = v. \end{cases}$$