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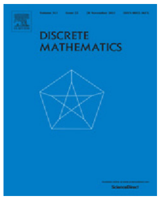


# On the diameter of cut polytopes

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ABSTRACT

Given an undirected graph $G$ with node set $V$ and edge set $E$, the cut polytope $CUT(G)$ is defined as the convex hull of the incidence vectors of the cuts in $G$. For a given integer $k \in \{1, 2, \ldots, \lfloor \frac{|V|}{2} \rfloor\}$, the uniform cut polytope $CUT_{=k}(G)$ is defined as the convex hull of the incidence vectors of the cuts which correspond to a bipartition of the node set into sets with cardinalities $k$ and $|V| - k$.

In this paper, we study the diameter of these two families of polytopes. For a connected graph $G$, we prove that the diameter of $CUT(G)$ is at most $|V| - 1$, improving on the bound of $|E|$ given by F. Barahona and A.R. Mahjoub. We also give its exact value for trees and complete bipartite graphs. Then, with respect to uniform cut polytopes, we introduce sufficient conditions for adjacency of two vertices in $CUT_{=k}(G)$, we study some particular cases and provide some connections with other partition polytopes from the literature.


## 1. Introduction

*Diameter of polytopes*

Given a nonempty polytope $P$, the 1-*skeleton* of $P$ is the graph whose nodes correspond to the vertices of $P$, and having an edge joining two nodes iff (if and only if) the corresponding vertices of $P$ are adjacent on $P$. Given two vertices $v_1, v_2$ of $P$, let $d(v_1, v_2)$ denote the length (w.r.t. the number of edges) of a shortest path between the nodes representing $v_1$ and $v_2$ in the 1-skeleton of $P$. The *diameter* of $P$, which is denoted by *diameter*$(P)$ is the maximum distance between any two vertices of $P$, i.e. $\max\{d(v_1, v_2) : v_1, v_2 \text{ are vertices of } P\}$.

The notion of diameter of a polyhedron presents, among others, some connections with linear programming and linear optimization methods that are based on the fact that if a linear program having for feasible region a pointed polyhedron has a finite optimal objective value, then there exists an optimal vertex solution. For example, consider the simplex algorithm applied to such a linear program: starting from a particular vertex $v_0$ of the feasible region $F$, this method consists in iteratively moving from a vertex (*the current basic feasible solution*) of $F$ to an adjacent vertex (with better or equal objective value), until an optimal vertex solution is found.

Given a nonempty polytope $P$, any vertex $v$ of $P$ is the unique minimizer of some linear function over $P$. For such an objective function, if the simplex method is applied starting from a vertex $v_0$ satisfying $d(v, v_0) = diameter(P)$, then the number of iterations required is at least *diameter*$(P)$.

For further information on polytopes and related notions, the reader may consult, e.g., the textbooks by Grünbaum [8] and Ziegler [16]. We now introduce some useful notation and preliminaries.

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*Notation and preliminaries*

Let $G = (V, E)$ denote an undirected graph with node set $V$ and edge set $E$. Given a node set $S \subseteq V$, let $\delta_G(S)$ (or $\delta(S)$ when $G$ is clear from the context) stand for the cut that is defined by $S$, i.e., the set of edges in $E$ with exactly one endpoint in $S$: $\delta(S) = \{e \in E : |e \cap S| = 1\}$. The node sets $S$ and $V \setminus S$ are called the *shores* of the cut $\delta(S)$. If $0 < |S| < |V|$, the cut $\delta(S)$ is said to be *proper*. The set of edges in the graph $G = (V, E)$ with both endpoints in $S \subseteq V$ is denoted by $E_G(S)$. If there exists an edge joining the nodes $u$ and $v$, i.e. $\{u, v\} \in E$, then the nodes $u$ and $v$ are said to be *adjacent*.

Given a set $F$ of edges in $G$, its incidence vector $\chi^F \in \mathbb{R}^{|E|}$ is defined by $\chi^F_e = 1$ if $e \in F$ and 0 otherwise. Given a node $v \in V$, $N(v)$ denotes the set of nodes in $V \setminus \{v\}$ that are adjacent to $v$ in $G$. Given $S \subseteq V$, $G[S]$ denotes the subgraph of $G$ that is induced by the node set $S$. Given a graph $G$, the notations $V(G)$ and $E(G)$ indicate the node set and the edge set of $G$, respectively.

A set $P$ of vectors in $\mathbb{R}^n$ is called a *polyhedron* if $P = \{x \in \mathbb{R}^n : Ax \le b\}$, for some matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$. A bounded polyhedron is also called a *polytope*. The dimension of a polyhedron $P$, denoted by $\dim(P)$, is the maximum number of affinely independent points in $P$ minus 1. Given a polyhedron $P \subset \mathbb{R}^n$, a set $F$ is a *face* of $P$ iff $F \neq \emptyset$ and $F = \{x \in P : \bar{A}x = \bar{b}\}$, for some subsystem $\bar{A}x \le \bar{b}$ of $Ax \le b$. A face of $P$ having dimension 0 (resp. 1, $\dim(P) - 1$) is also called a *vertex* (resp. an *edge*, a *facet*) of $P$. Two vertices of a polyhedron $P$ are called *adjacent* or *neighbouring* if they are both contained in an edge of $P$.

Given two sets $A$ and $B$, the notation $A \Delta B$ represents their symmetric difference, i.e. $A \Delta B = (A \cup B) \setminus (A \cap B)$. For any node subsets $T, U \subseteq V$, $\delta(T)\Delta\delta(U) = \delta(T \Delta U)$, hence the collection of cuts is closed under taking symmetric differences (see for example [15]). A cut $I$ is said to be *minimal* if there does not exist any cut $J \subsetneq I$.

*On the cut and uniform cut polytopes*

The *cut polytope* $CUT(G)$ is defined as the convex hull of the incidence vectors of all the cuts in the graph $G$, i.e. $CUT(G) = \text{conv}\{\chi^{\delta_G(S)} : S \subseteq V\}$. Given some integer $k$ satisfying $k \le \lfloor \frac{|V|}{2} \rfloor$, $CUT_{=k}(G) = \text{conv}\{\chi^{\delta_G(S)} : S \subseteq V \text{ and } |S| = k\}$ denotes the convex hull of the incidence vectors of the cuts in $G$ having one shore with size $k$. Polytopes of the latter form are called *uniform cut polytopes*. In the particular case when the graph $G$ is complete with order $n$, i.e., $G = K_n$, we shall simply write: $CUT^n_{=k}$.

Investigations on the polyhedral structure of these and related polytopes are reported, e.g., in [2,6,13,14] and the references therein. Optimization over the cut polytope arises, e.g., in statistical physics [1,5] or circuit design [7] (for further information on properties and applications of these families of polytopes the reader may consult, e.g., [6]). Our objective is to develop new insights and improve our knowledge of the polyhedral structure of these and related polytopes.

To the author's knowledge, the first study of the 1-skeleton of the cut polytope appears in the paper by Barahona and Mahjoub [2], where the following characterization of vertex adjacency on the cut polytope is provided (Theorem 4.1 in [2]).

**Theorem 1.1** ([2]). *Let $G = (V, E)$ be a connected graph, and let $I, J$ denote two cuts in $G$. Set $F = E \setminus (I \Delta J)$. Then $\chi^I$ and $\chi^J$ are adjacent on $CUT(G)$ iff the graph $H_{I\Delta J} = (V, F)$ has two connected components.* □

The proof the authors give relies on the property that the extreme points $\chi^I$ and $\chi^J$ are adjacent on $CUT(G)$ iff there exists a vector $c$ such that they are the only two vertices optimizing $\max\{c^T x : x \in CUT(G)\}$. The following remark directly follows from Theorem 1.1.

**Remark 1.** The vectors $\chi^I$ and $\chi^J$ are adjacent on $CUT(G)$ iff $I \Delta J$ is a minimal cut. Indeed, assuming that $I \Delta J$ is not a minimal cut, let $L$ denote a minimal cut such that $\emptyset \neq L \subsetneq I \Delta J$. Then $\chi^{I\Delta L} + \chi^{J\Delta L} = \chi^I + \chi^J$, implying that $\chi^I$ and $\chi^J$ cannot be adjacent. Conversely, if we now assume that $I \Delta J$ is a minimal cut, then there cannot be more than two connected components in $H_{I\Delta J}$ because otherwise any node set corresponding to a connected component in $H_{I\Delta J}$ would define a nonempty cut strictly contained in $I \Delta J$.

In the same reference [2], it is also established that the Hirsch property holds for $CUT(G)$. (A $d$-dimensional polytope $P$ with $f$ facets is said to satisfy the Hirsch property if its diameter verifies the inequality: $diameter(P) \le f - d$.) In fact, Naddef [11] proved later that this property holds for all (0, 1)-polytopes, and thus also for $CUT(G)$ and the uniform cut polytopes. What also follows from Theorem 1.1 and its proof is that $CUT(G)$ is a so-called *combinatorial polyhedron* [10,12], a family of polyhedra for which Matsui and Tamura established further different properties related to the Hirsch property [10].

The present paper is organized as follows. In Section 2, we study the diameter of the cut polytope, give its exact value for some graph classes and provide upper bounds for other cases. In Section 3 we report investigations on the diameter of uniform cut polytopes: after we provide (Section 3.1) sufficient conditions for two vertices to be adjacent on these polytopes, we deal with some particular cases, drawing some connections with other polytopes from the literature (Section 3.2).