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Decomposition of the Kneser Graph into paths of length four

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ABSTRACT

Necessary and sufficient conditions are given for the existence of a graph decomposition of the Kneser Graph $KG_{n,2}$ into paths of length four.

1. Introduction

One of the more widely studied topics in the intersection of design and graph theory is the topic of graph decompositions. An H-decomposition of a graph $G = (V, E)$ is a pair (V, B), where B is a collection of edge-disjoint subgraphs of G, each isomorphic to H, whose edges partition $E(G)$. Many different combinations of H and G have been studied, but certain ubiquitous graphs have been studied quite thoroughly. For example, the subgraphs used to partition the edges of G are frequently complete graphs [9,11], cycles [1,10,16], paths [18], or spanning trees [5]. In these papers, G is a complete graph or a complete multipartite graph, possibly with the edges in a 1-factor removed. Other possibilities for G have also been considered, such as the multigraph formed from a complete multipartite graph by joining vertices in the same part with λ edges (see [6,7], for example). The focus of this paper is on the case where G is the Kneser Graph.

The Kneser Graph $KG_{n,k}$ is the graph whose vertices are the k-element subsets of some set of n elements, in which two vertices are adjacent if and only if their intersection is empty. This family of graphs first appeared in the literature in 1955 [12]. Two of the more studied problems surrounding the Kneser Graphs are to find their chromatic number and to determine for which n and k they are Hamiltonian [4,17]. Kneser's original conjecture that $\chi(KG_{n,k}) = n - 2k + 2$ whenever $n \geq 2k$ has resulted in several significant developments in the field [2,8,12–14]. In particular, the proof by Lovász [13] in 1978 using topological methods, namely the Borsuk–Ulam theorem, contributed to the rise of interest in topological combinatorics.

Contrary to the usual notation, because the number of edges in the path is of primary interest in this paper, here P_i is defined to be a path of *length i*. Necessary and sufficient conditions for the existence of P_3-decompositions of $KG_{n,2}$ were found in [15]. In [15], the existence of P_3-decompositions of the Generalized Kneser Graph $GKG_{n,k,r}$ (the graph whose vertices are the k-element subsets of some set of n elements in which two vertices are adjacent if and only if they intersect in precisely r elements) was also settled. The purpose of this paper is to establish necessary and sufficient conditions for the existence of P_4-decompositions of $KG_{n,2}$. This problem is completely solved in Theorem 1, where an explicit construction is given.

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2. Useful building blocks

Let $T(V)$ be the set of 2-element subsets of the set V and as usual let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$.Let (a, b, c, d, e) denote the path, P_4, of length 4 induced by the four edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, e\}$.

The following useful result was proved in more generality by Billington and Hoffman [3], but the following will suffice for our purposes.

Lemma 1. *The complete bipartite graph K_{a_1,a_2} with $a_1 \le a_2$ has a P_4-decomposition if and only if $a_1 \ge 2$, $a_2 \ge 3$, and $a_1a_2 \equiv 0$ (mod 4).*

The next result provides specific ingredients used in the general constructions.

Lemma 2. *There exists a P_4-decomposition of:*

(i) *the bipartite graph H_1 with bipartition $\{A, B\}$ of $V(H_1)$, with $A = T(\mathbb{Z}_4)$ and $B = \mathbb{Z}_4$, and $E(H_1) = \{\{a, b\} | a \in A, b \in B, b \notin a\}$,*
(ii) *the bipartite graph H_2 with bipartition $\{A, B\}$ of $V(H_2)$, with $A = T(\mathbb{Z}_6)$ and $B = \mathbb{Z}_6$, and $E(H_2) = \{\{a, b\} | a \in A, b \in B, b \notin a\}$,*
(iii) *$H_3(W, X, Y) = (W \cup X \cup Y, E)$, with $W = \{w_1, w_2, w_3, w_4\}$, $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ being disjoint sets of size 4, and $E = \{\{(w_l, i_j)\} \mid w_l \in W, i_j \in X \cup Y, l \ne j\}$,*
(iv) *$H_4 = (\mathbb{Z}_4 \times \mathbb{Z}_4, \{(i, j), (k, l) \mid i \ne k, j \ne l\})$,*
(v) *$H_5(W, X, Y, Z) = (W \cup X \cup Y \cup Z, E)$, with $W = \{w_1, w_2, w_3, w_4\}$, $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $Z = \{z_1, z_2, z_3, z_4\}$ being disjoint sets of size 4, and $E = \{\{(w_l, i_j)\} \mid w_l \in W, i_j \in X \cup Y \cup Z, l \ne j\}$,*
(vi) *the bipartite graph H_6 with bipartition $\{A, B\}$ of $V(H_6)$, where $A = T(\mathbb{Z}_5) \cup T(\mathbb{Z}_9 \backslash \mathbb{Z}_5)$ and $B = \mathbb{Z}_5$, and $E(H_6) = \{\{a, b\} | a \in A, b \in B, b \notin a\}$, and*
(vii) *K_8.*

Proof. (i) $(V(H_1), \{(0, \{1, 2\}, 3, \{0, 1\}, 2), (3, \{0, 2\}, 1, \{2, 3\}, 0), (0, \{1, 3\}, 2, \{0, 3\}, 1)\})$ is the required decomposition.

(ii) Let $B_1 = \{(1 + 3i, \{3i, 2 + 3i\}, 5 + 3i, \{1 + 3i, 2 + 3i\}, 4 + 3i), (3 + 3i, \{3i, 2 + 3i\}, 4 + 3i, \{3i, 1 + 3i\}, 5 + 3i), (2 + 3i, \{3i, 1 + 3i\}, 3 + 3i, \{1 + 3i, 2 + 3i\}, 3i) \mid i \in \{0, 1\}\}$ with addition done modulo 6, and let $B_2 = \{(j + 1, \{j, 3\}, 4, \{j, 5\}, j + 2), (\{j, 3\}, j + 2, \{j, 4\}, 3, \{j, 5\}), (\{j, 3\}, 5, \{j, 4\}, j + 1, \{j, 5\}) \mid j \in \{0, 1, 2\}\}$ with addition done modulo 3 (it is important to note that in this case each sum is in $\mathbb{Z}_3$, not $\mathbb{Z}_6$). Then $(V(H_2), B_1 \cup B_2)$ is the required decomposition.

(iii) $(W \cup X \cup Y, \{(x_1, w_3, x_4, w_1, x_3), (x_2, w_4, x_3, w_2, x_4), (y_1, w_3, y_2, w_1, y_4), (y_2, w_4, y_1, w_2, y_3), (w_1, x_2, w_3, y_4, w_2), (w_1, y_3, w_4, x_1, w_2)\})$ is the required decomposition.

(iv) The result follows from (iii), since H_4 is the union of the three graphs $H_3(\mathbb{Z}_4 \times \{i\}, \mathbb{Z}_4 \times \{j\}, \mathbb{Z}_4 \times \{k\})$, where $(i, j, k) \in \{(1, 0, 2), (3, 2, 1), (0, 3, 2)\}$.

(v) $(W \cup X \cup Y \cup Z, \{(x_0, w_2, x_1, w_0, x_2), (x_1, w_3, x_2, w_1, x_3), (y_0, w_1, y_2, w_0, y_1), (y_3, w_2, y_1, w_3, y_2), (z_1, w_0, z_2, w_1, z_3), (z_2, w_3, z_1, w_2, z_3), (z_3, w_0, y_3, w_1, z_0), (w_2, z_0, w_3, x_0, w_1), (w_0, x_3, w_2, y_0, w_3)\})$ is the required decomposition.

(vi) $(V(H_6), \{(\{0, 1\}, 3, \{0, 2\}, 4, \{5, 6\}), (\{0, 2\}, 1, \{0, 3\}, 4, \{5, 7\}), (\{0, 3\}, 2, \{0, 4\}, 1, \{6, 8\}), (\{0, 4\}, 3, \{1, 2\}, 4, \{5, 8\}), (\{1, 2\}, 0, \{1, 3\}, 4, \{6, 7\}), (\{1, 3\}, 2, \{1, 4\}, 3, \{6, 8\}), (\{1, 4\}, 0, \{2, 3\}, 4, \{6, 8\}), (\{2, 3\}, 1, \{2, 4\}, 3, \{7, 8\}), (\{2, 4\}, 0, \{3, 4\}, 1, \{7, 8\}), (\{3, 4\}, 2, \{0, 1\}, 4, \{7, 8\}), (\{5, 6\}, 1, \{5, 7\}, 3, \{5, 8\}), (\{5, 6\}, 3, \{6, 7\}, 1, \{5, 8\}), (\{5, 6\}, 0, \{5, 7\}, 2, \{5, 8\}), (\{5, 6\}, 2, \{6, 7\}, 0, \{6, 8\}), (\{6, 8\}, 2, \{7, 8\}, 0, \{5, 8\})\})$ is the required decomposition.

(vii) Define K_8 on the vertices $\mathbb{Z}_7 \cup \{\infty\}$. Then $(\mathbb{Z}_7 \cup \{\infty\}, \{(\infty, i, i + 1, i - 1, i + 2) \mid i \in \mathbb{Z}_7\})$ with addition done modulo 7 is the required decomposition. □

Lemma 3. *$KG_{16,2}$ is P_4-decomposable.*

Proof. Consider $G = KG_{16,2}$ on vertex set $T(\mathbb{Z}_{16})$. Partition $\mathbb{Z}_{16}$ into four sets $S_i = \{4i, 4i + 1, 4i + 2, 4i + 3\}$ for $i \in \mathbb{Z}_4$. Partition the set of vertices $T(\mathbb{Z}_{16})$ of G into the following two types:

Type 1: $V_i = \{\{x, y\} \mid x, y \in S_i, x \ne y\}$ for each $i \in \mathbb{Z}_4$, and

Type 2: $V_{i,j} = \{\{x, y\} \mid x \in S_i, y \in S_j\}$ for $0 \le i < j < 4$.

First consider the subgraph G' of G induced by the Type 1 vertices. Decompose G' into paths of length 4 in two steps. First, for each $i \in \mathbb{Z}_4$ define three pairs of vertices $M_{0,i} = \{\{4i, 4i + 1\}, \{4i + 2, 4i + 3\}\}$, $M_{1,i} = \{\{4i, 4i + 2\}, \{4i + 1, 4i + 3\}\}$, and $M_{2,i} = \{\{4i, 4i + 3\}, \{4i + 1, 4i + 2\}\}$. For each $j \in \mathbb{Z}_3$, the subgraph G'_j of G' induced by $\bigcup_{i\in\mathbb{Z}_4} M_{j,i}$ is isomorphic to K_8 which therefore has a P_4-decomposition $(\bigcup_{i\in\mathbb{Z}_4} M_{j,i}, B'_j)$ by Lemma 2(vii). Let $B_1 = \bigcup_{j\in\mathbb{Z}_3} B'_j$. Second, for $0 \le i_1 < i_2 \le 3$ and for $j \in \mathbb{Z}_3$, the induced bipartite subgraph $G'_{i_1,i_2,j}$ of G' with bipartition $\{M_{j,i_1}, \bigcup_{k\in\mathbb{Z}_3\setminus\{j\}} M_{k,i_2}\}$ is isomorphic to $K_{2,4}$ and so has a P_4-decomposition $(V(G'_{i_1,i_2,j}), B'_{i_1,i_2,j})$ by Lemma 1. Let $B_2 = \bigcup_{0\le i_1<i_2\le 3, j\in\mathbb{Z}_3} B'_{i_1,i_2,j}$.

All edges connecting Type 1 vertices have now been placed into paths in $B_1 \cup B_2$. The remaining edges are those connecting Type 2 vertices and those connecting a Type 1 vertex to a Type 2.

The subgraph $G_{i,j}$ of G induced by the vertices in $V_{i,j}$ for $0 \le i < j < 4$ is isomorphic to H_4, so has a P_4-decomposition $(V(G_{i,j}), B_{i,j})$ by Lemma 2(iv). Let $B_3 = \bigcup_{0\le i<j<4} B_{i,j}$.