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Spanning trees and spanning Eulerian subgraphs with small degrees

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ABSTRACT

Liu and Xu (1998) and Ellingham, Nam and Voss (2002) independently showed that every k-edge-connected simple graph G has a spanning tree T such that for each vertex v, $d_T(v) \leq \lceil \frac{d(v)}{k} \rceil + 2$. In this paper we show that every k-edge-connected graph G has a spanning tree T such that for each vertex v, $d_T(v) \leq \lceil \frac{d(v)-2}{k} \rceil + 2$; also if G has k edge-disjoint spanning trees, then T can be found such that for each vertex v, $d_T(v) \leq \lceil \frac{d(v)-1}{k} \rceil + 1$. This result implies that every $(r-1)$-edge-connected r-regular graph (with $r \geq 4$) has a spanning Eulerian subgraph whose degrees lie in the set $\{2, 4, 6\}$; also reduces the edge-connectivity needed for some theorems due to Barát and Gerbner (2014) and Thomassen (2008, 2013). Moreover these bounds for finding spanning trees are sharp.

1. Introduction

In this article, all graphs have no loop, but multiple edges are allowed and a simple graph is a graph without multiple edges. All integer variables are positive, unless otherwise stated. Let G be a graph. The vertex set, the edge set and the maximum degrees of vertices of G are denoted by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. Let f be a positive integer-valued function on $V(G)$, a spanning tree T of G is called **spanning** f**-tree**, if for each vertex v, $d_T(v) \leq f(v)$. For a set A of integers, an A-**factor** is a spanning subgraph with vertex degrees in A. Let K be a subgraph of G and $S \subseteq V(G)$, the set of edges of K whose ends are in the set S, denoted by $E_S(K)$.

In 1997, Czumaj and Strothmann used the algorithm in [9] (when the input graph is k-connected) to deduce the following theorem.

Theorem 1 ([6]). *Every k-connected graph G has a spanning tree T such that $\Delta(T) \leq \lceil \frac{\Delta(G)-2}{k} \rceil + 2$.*

Liu and Xu (1998) investigated spanning trees with small degrees in highly edge-connected simple graphs. They concluded that every r-edge-connected r-regular simple graph has a connected $\{1, 2, 3\}$-factor and remarked that these graphs may have no connected $\{1, 2\}$-factors (Hamiltonian path). These results were rediscovered independently by Ellingham, Nam and Voss (2002).

Theorem 2 ([7,11]). *Every k-edge-connected simple graph G has a spanning tree T such that for each vertex v, $d_T(v) \leq \lceil \frac{d(v)}{k} \rceil + 2$.*

In the special case $k = 2$, the upper bound can be reduced. Many authors found the following fact with different proofs. In addition, Theorem 3 was discovered in [6], but for 2-connected graphs.

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Theorem 3 *([1,11,14]). Every 2-edge-connected graph G has a spanning tree T such that for each vertex v, $d_T(v) \leq \lceil \frac{d(v)}{2} \rceil + 1$.*

Let G be a k-edge-connected graph. By the same fundamental process in [11], we show that in Section 2, the graph G has a spanning tree T such that for each vertex v, $d_T(v) \leq \lceil \frac{d(v)-2}{k} \rceil + 2$. This improves the results of Theorems 1 and 2 and also implies Theorem 3. Accordingly, this shows that every $(r-2)$-edge-connected r-regular graph (with $r \geq 3$) has a connected $\{1, 2, 3\}$-factor.

The authors of [2,14,15] confirmed the conjecture of Barát–Thomassen [3] for some trees, using the fact that every graph G with k edge-disjoint spanning trees has a spanning tree with small degrees. In Section 3, we prove that G has a spanning tree T such that for each vertex v, $d_T(v) \leq \lceil \frac{d(v)-1}{k} \rceil + 1$. This improved bound reduces the edge-connectivity needed for some theorems of above mentioned papers, for example, reduces the edge-connectivity of the following theorem to 75 with exactly the same proof. Note that Martin Merker (pers. comm.) can also decrease the necessary edge-connectivity less than 60.

Theorem 4 *([14]). Every 171-edge-connected simple graph has an edge-decomposition into paths of length 3 if and only its size is divisible by 3.*

Jaeger [10] and Catlin [5] independently showed that every 4-edge-connected graph has a spanning Eulerian subgraph. In Section 4, we show that every $(r-1)$-edge-connected r-regular graph (with $r \geq 4$) has a spanning Eulerian subgraph whose degrees lie in the set $\{2, 4, 6\}$. Note that for a 4-connected $K_{1,3}$-free graph, this set can be replaced by $\{2, 4\}$, which is proved in [4].

2. Spanning trees in graphs with high edge-connectivity

Liu and Xu [11] used the idea of Lemma 1 and 2 in [17] and generalized them to prove Theorem 2. Also Ellingham and Zha [8] found a shorter and more systematic proof for them. We state the following important lemma, which generalizes Lemma 1 in [11].

Lemma 1. *Let G be a connected graph with a positive integer-valued function f on $V(G)$. If G has no spanning f-tree, then there exists a proper induced subgraph H of G, a spanning f-tree T of H and a non-empty subset S of $V(T)$, with the following properties:*

1. *For each vertex v of S, $d_T(v) = f(v)$.*
2. *There is no edge of H joining components of $T \setminus S$.*
3. *The set S contains all vertices of H which are adjacent to a vertex in $V(G) \setminus V(H)$.*

Proof. Let H be a proper induced subgraph of G with a spanning f-tree T. Consider H with maximum $|V(H)|$. For any $S \subseteq V(H)$ and $v \in V(H) \setminus S$, let $\mathcal{A}(S, v)$ be the set of all spanning f-trees T' of H such that T and T' have the same edges, except for some edges whose ends are in C, where C is the component of $T \setminus S$ containing v. Let $V_0 = \emptyset$ and for each positive integer i, recursively define V_i as follows:

$$V_i = \{v \in V(H) \colon d_{T'}(v) = f(v), \text{ for all } T' \in \mathcal{A}(V_0 \cup \cdots \cup V_{i-1}, v)\}.$$

Now we prove the following claim.

Claim. *Let u and v be two vertices in different components of $T \setminus (V_0 \cup \cdots \cup V_{n-1})$. If $uv \in E(H)$, then $u \in V_n$ or $v \in V_n$.*

Proof of Claim. By induction on n. For $n = 1$, the proof is clear. Assume that the claim is true for $n - 1$. Now we prove it for n. Suppose to the contrary that vertices u and v are in different components of $T \setminus (V_0 \cup \cdots \cup V_{n-1})$, $uv \in E(H)$ and $u, v \notin V_n$. Then there exist $T_1 \in \mathcal{A}(V_0 \cup \cdots \cup V_{n-1}, v)$ and $T_2 \in \mathcal{A}(V_0 \cup \cdots \cup V_{n-1}, u)$ with $d_{T_1}(v) \neq f(v)$ and $d_{T_2}(u) \neq f(u)$. Note that T_1 and T_2 are spanning f-trees of H. Let P be the unique path connecting u and v in T. By the induction hypothesis, u and v are in the same component of $T \setminus (V_0 \cup \cdots \cup V_{n-2})$. Hence $V(P) \cap (V_0 \cup \cdots \cup V_{n-2}) = \emptyset$. This implies that $V(P) \cap V_{n-1} \neq \emptyset$. Pick $w \in V(P) \cap V_{n-1}$. Let e be an edge of P adjacent to w. Now let $\mathcal{T}$ be a spanning tree of H with

$$E(\mathcal{T}) = D(v, T_1) \cup D(u, T_2) \cup \{uv\} \cup \big(E(T) \setminus \big(D(v, T) \cup D(u, T) \cup \{e\}\big)\big),$$

where $D(x, T')$ is the set of edges of $T' \setminus (V_0 \cup \cdots \cup V_{n-1})$ whose ends are in the component containing x, for any $x \in \{u, v\}$ and any $T' \in \{T, T_1, T_2\}$. It is not hard to see that $\mathcal{T}$ lies in $\mathcal{A}(V_0 \cup \cdots \cup V_{n-2}, w)$ and $d_{\mathcal{T}}(w) < f(w)$. But $w \in V_{n-1}$, which is a contradiction. Hence the claim holds.

Since $V_0, V_1, \ldots$ are pairwise disjoint sets, there exists a positive integer k with $V_k = \emptyset$. Let $S = V_0 \cup \cdots \cup V_{k-1}$. For each $v \in V_i$, we have $T \in \mathcal{A}(V_0 \cup \cdots \cup V_{i-1}, v)$ and so $d_T(v) = f(v)$. This establishes Condition 1. Because $V_k = \emptyset$, the previous claim implies Condition 2. Since f is positive, by considering the construction of H, for each $T' \in \mathcal{A}(\emptyset, u)$, $d_{T'}(u) = f(u)$, where u is a vertex of H adjacent to a vertex in $V(G) \setminus V(H)$. Thus V_1 contains all vertices of H adjacent to a vertex in $V(G) \setminus V(H)$. So Condition 3 holds. Since G is connected, S is non-empty. Therefore the lemma is proved. □

A special case of the next lemma appeared in [11].