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On three sets with nondecreasing diameter

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ABSTRACT

Let $[a, b]$ denote the integers between a and b inclusive and, for a finite subset $X \subseteq \mathbb{Z}$, let $\mathrm{diam}\,(X) = \max(X) - \min(X)$. We write $X <_p Y$ provided $\max(X) < \min(Y)$. For a positive integer m, let $f(m, m, m; 2)$ be the least integer N such that any 2-coloring $\Delta : [1, N] \to \{0, 1\}$ has three monochromatic m-sets $B_1, B_2, B_3 \subseteq [1, N]$ (not necessarily of the same color) with $B_1 <_p B_2 <_p B_3$ and $\mathrm{diam}\,(B_1) \le \mathrm{diam}\,(B_2) \le \mathrm{diam}\,(B_3)$. Improving upon upper and lower bounds of Bialostocki, Erdős and Lefmann, we show that $f(m, m, m; 2) = 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta$ for $m \ge 2$, where $\delta = 1$ if $m \in \{2, 5\}$ and $\delta = 0$ otherwise.

1. Introduction

For $a, b \in \mathbb{R}$, we let $[a, b]$ denote the set of integers between a and b inclusive. For finite subsets $X, Y \subseteq \mathbb{Z}$, the *diameter* of X, denoted by $\mathrm{diam}\,(X)$, is defined as $\max(X) - \min(X)$. Moreover, we say that $X <_p Y$ if and only if $\max(X) < \min(Y)$, meaning all the elements of X come before any element from Y. For positive integers $t, m_1, m_2, \ldots, m_t, r$, let $f(m_1, m_2, \ldots, m_t; r)$ be the least integer N such that, for every r-coloring $\Delta : [1, N] \to [0, r-1]$ of the integers $[1, N]$, there exist t subsets $B_1, B_2, \ldots, B_t \subseteq [1, N]$ with

(a) each B_i monochromatic, i.e., $|\Delta(B_i)| = 1$ for $i = 1, \ldots, t$,
(b) $|B_i| = m_i$ for $i = 1, 2, \ldots, t$
(c) $B_1 <_p B_2 <_p \cdots <_p B_t$, and
(d) $\mathrm{diam}\,(B_1) \le \mathrm{diam}\,(B_2) \le \cdots \le \mathrm{diam}\,(B_t)$.

A collection of sets B_i that satisfy (a), (b), (c) and (d) is called a *solution* to the problem defined by $p(m_1, m_2, \ldots, m_t; r)$.

The function $f(m_1, m_2, \ldots, m_t; r)$, and the related function $f^*(m_1, m_2, \ldots, m_t; r)$ defined as $f(m_1, m_2, \ldots, m_t; r)$ but requiring the inequalities in (d) to be strict, have been studied by previous authors. Bialostocki, Erdős and Lefmann first introduced $f(m, m, \ldots, m; r)$ in [3], where they determined that $f(m, m; 2) = 5m - 3$, that $f(m, m; 3) = 9m - 7$ and that

$$8m - 4 \le f(m, m, m; 2) \le 10m - 6, \tag{1}$$

as well as giving asymptotic bounds for $t = 2$. The problem was motivated in part by zero-sum generalizations in the sense of the Erdős–Ginzburg–Ziv Theorem [6], [9, Theorem 10.1] (see [2,3,7] for a short discussion of zero-sum generalizations,

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including definitions). Subsequently, Bollobás, Erdős, and Jin [4] obtained improved results for $m = 2$, showing that $4r - \log_2 r + 1 \leq f^*(2, 2; r) \leq 4r + 1$ and $f^*(2, 2; 2^k) = 4 \cdot 2^k + 1$, as well as giving improved asymptotic bounds for t and r when $m = 2$. The value of $f(m, m; 4)$ was determined to be $12m - 9$ in [8], the off-diagonal cases (when not all $m_i = m$) are introduced in [14], and other related Ramsey-type problems can also be found in [1,5,10,12,11,13].

The goal of this paper is to improve the estimates from (1) to the first exact value for more than two sets. Indeed, we will show that both the upper and lower bounds of Bialostocki, Erdős and Lefmann can be improved, resulting in the value

$$f(m, m, m; 2) = 8m - 5 + \left\lfloor \frac{2m-2}{3} \right\rfloor + \delta \quad \text{for } m \geq 2,$$

where $\delta = 1$ if $m \in \{2, 5\}$ and $\delta = 0$ otherwise.

2. Determination of $f(m, m, m; 2)$

Let $\Delta : X \to C$ be a finite coloring of a finite set X by a set of colors C. Let $c \in C$ and $Y \subseteq X$. Let $x_1 < x_2 < \cdots < x_n$ be the integers colored by c in Y. Then, for integers i and j such that $1 \leq i \leq j \leq n$, we use the notation $\mathsf{first}_i^j(c, Y)$ to denote $\{x_i, x_{i+1}, \ldots, x_j\}$, which is the set consisting of the ith through jth smallest elements of Y colored by c. Likewise, $\mathsf{first}_i(c, Y) = x_i$ is the ith smallest element colored by c in Y, and $\mathsf{first}(c, Y) = \mathsf{first}_1(c, Y)$ is the first element colored by c in Y. Similarly, we define $\mathsf{last}_i^j(c, Y) = \{x_{n-i+1}, x_{n-i}, \ldots, x_{n-j+1}\}$ to be the set consisting of the ith through jth largest elements of Y colored by c, $\mathsf{last}_i(c, Y) = x_{n-i+1}$ to be the ith largest element of Y colored by c, and $\mathsf{last}(c, Y) = \mathsf{last}_1(c, Y)$ to be the largest element of Y colored by c. For the sake of simplicity, a coloring $\Delta : [1, N] \to C$ will be denoted by the string $\Delta(1)\Delta(2)\Delta(3)\cdots\Delta(N)$, and x^i will be used to denote the string $xx\ldots x$ of length i. Hence $\Delta : [1, 6] \to \{0, 1\}$, where $\Delta([1, 2]) = \{0\}$, $\Delta(3) = 1$, and $\Delta([4, 6]) = \{0\}$, may be represented by the string $\Delta[1, 6] = 0^2 10^3$.

The following technical lemma will help us control the possible 2-colorings of $[1, 3m - 2]$.

Lemma 2.1. *Let $m \geq 2$, let $\Delta : [1, 3m - 2] \to \{0, 1\}$ be a 2-coloring and let $B_1 \subseteq [1, 3m - 2]$ be a monochromatic m-subset with $\mathsf{diam}\,(B_1) \geq 2m - 2$ satisfying the following additional extremal constraints:*

(a) $\max B_1 := 3m - 2 - \beta$ *is minimal, where* $\beta \in [0, m - 1]$*;*
(b) $\mathsf{diam}\,(B_1) := 2m - 2 + \alpha$ *is minimal subject to* (a) *holding, where* $\alpha \in [0, m - 1 - \beta]$.

Suppose B_1 exists and $\Delta(B_1) = \{1\}$. Then, letting $R = [1, 3m - 2 - \beta]$, one of the following holds.

(i)
- $\beta \leq m - 2$ *and* $|\Delta^{-1}(0) \cap R| \geq m$
- $\Delta R = 1^{m-1-\beta-\nu} H_0 0 H_1 1^{1+\nu}$, *where* $\mu,\ \nu \geq 0$ *are integers*
- $\beta = m - 1 - \alpha$ *or* $\nu = \mu = 0$
- H_1 *is a string of length* $m - 2 - \beta + \mu$ *with exactly* μ *1's and exactly* $m - 2 - \beta$ *0's*
- H_0 *is a string of length* $m - 1 + \beta - \mu$ *containing exactly* $m - 1 - \alpha - \mu$ *1's and exactly* $\alpha + \beta$ *0's*

(ii)
- *either* $\beta < m - 1 - \alpha$ *or* $\beta = m - 1$
- $\Delta R = 0^{m-\alpha-\beta-1} 1 H_2 1^{m-\beta}$
- H_2 *is a string of length* $m - 2 + \beta + \alpha$
- *if* $\beta \leq m - 2$, *then* $|\Delta^{-1}(0) \cap R| \geq m$
- *if* $\alpha > 0$, *then* $\beta \geq 1$ *and* H_2 *contains exactly* $\beta - 1$ *1's*

(iii)
- $\beta \geq \alpha$
- $|\Delta^{-1}(0) \cap R| < m$
- $\mathsf{first}_m(1, R) \leq 3m - 3 - \beta - \alpha$
- $|\Delta^{-1}(0) \cap [\mathsf{first}_1(1, R), \mathsf{first}_m(1, R)]| \leq \beta$.

Proof. Note (a) and (b) imply

$$\min B_1 = \max B_1 - \mathsf{diam}\, B_1 = 3m - 2 - \beta - (2m - 2 + \alpha) = m - \alpha - \beta. \tag{2}$$

Let

$$\eta = 3m - 3 - \beta - \mathsf{last}_2(1, R) \geq 0$$

be the number of integers colored by 0 between $\mathsf{last}_2(1, R)$ and $\mathsf{last}_1(1, R)$. Let

$$\nu = 3m - 3 - \beta - \mathsf{last}(0, R) \geq 0$$

be the number of integers strictly between $\mathsf{last}(0, R)$ and $3m - 2 - \beta$ that are colored by 1. We continue with three claims.

Claim A. *If $|\Delta^{-1}(1) \cap R| > m$, then $\mathsf{last}_2(1, R) \leq 3m - 3 - \beta - \alpha$ and $\eta \geq \alpha$.*

If we have $\mathsf{last}_2(1, R) \geq 2m - 2 + \mathsf{first}(1, R)$, then $B_1' = \mathsf{first}_1^{m-1}(1, R) \cup \{\mathsf{last}_2(1, R)\}$ will be a monochromatic m-subset, in view of the hypothesis $|\Delta^{-1}(1) \cap R| > m$, with $\max B_1' < \max B_1$ and $\mathsf{diam}\, B_1' = \mathsf{last}_2(1, R) - \mathsf{first}(1, R) \geq 2m - 2$, contradicting the maximality condition (a) for B_1. Therefore we may instead assume $\mathsf{last}_2(1, R) \leq 2m - 3 + \mathsf{first}(1, R) \leq 3m - 3 - \beta - \alpha$, where the final inequality follows from $\mathsf{first}(1, R) \leq \min B_1$ and (2), and now $\eta \geq \alpha$ follows from the definition of η, completing the claim.