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Decomposition of the complete bipartite multigraph into cycles and stars

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ABSTRACT

Let C_k denote a cycle of length k, and let S_k denote a star with k edges. For multigraphs F, G, and H, a decomposition of F is a set of edge-disjoint subgraphs of F whose union is F, and a (G, H)-decomposition of F is a decomposition of F into copies of G and H using at least one of each. In this paper, necessary and sufficient conditions for the existence of a (C_k, S_k)-decomposition of the complete bipartite multigraph are given.

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1. Introduction

For positive integers m and n, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. A k-cycle, denoted by C_k, is a cycle of length k. A k-star, denoted by S_k, is the complete bipartite graph $K_{1,k}$. A *k-path*, denoted by P_k, is a path with k vertices. For a graph H and a positive integer λ, we use λH to denote the multigraph obtained from H by replacing each edge e by λ edges each having the same endpoints as e.

Let F, G, and H be multigraphs. A *decomposition* of F is a set of edge-disjoint subgraphs of F whose union is F. A G-*decomposition* of F is a decomposition of F into copies of G. If F has a G-decomposition, we say that F is G-*decomposable* and write $G|F$. A (G, H)-*decomposition* of F is a decomposition of F into copies of G and H using at least one of each. If F has a (G, H)-decomposition, we say that F is (G, H)-*decomposable* and write $(G, H)|F$. When $G_1, \ldots, G_t$ are multigraphs, not necessarily disjoint, we write $G_1 \cup \cdots \cup G_t$ or $\bigcup_{i=1}^{t} G_i$ for the graph with vertex set $\bigcup_{i=1}^{t} V(G_i)$ and edge set $\bigcup_{i=1}^{t} E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^{t} G_i$ expresses the decomposition of G into $G_1, \ldots, G_t$.

A great deal of work has been done on G-decomposition of graphs; see survey articles [8,9,12,31] and the book [10]. In particular, decomposition into k-cycles has attracted considerable attention; see [11,14,19] for surveys of this topic. Decomposition into k-stars has also attracted a fair share of interest; see [13,18,29,30,32,33]. It is natural to consider the problem of decomposing a graph into copies of distinct graphs. The study of (G, H)-decomposition was introduced by Abueida and Daven in [2]. Abueida and Daven [3] investigated the problem of (K_k, S_k)-decomposition of the complete graph K_n. Abueida and Daven [4] investigated the problem of the (C_4, M_2)-decomposition of several graph products, where M_2 denotes the 4-vertex graph having two disjoint edges. Abueida and O'Neil [7] settled the existence problem for (C_k, S_{k-1})-decomposition of the complete multigraph λK_n for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [21,22] gave necessary and sufficient conditions for the existence of (G_n, H_n)-decompositions of λK_n and $\lambda K_{n,n}$, where $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$.

A *graph-pair* (G, H) *of order* m is a pair of non-isomorphic graphs G and H on m non-isolated vertices such that $G \cup H$ is isomorphic to K_m. Abueida and Daven [2] and Abueida, Daven, and Roblee [5] completely determined the values of n

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for which λK_n admits a (G, H)-decomposition, where (G, H) is a graph-pair of order 4 or 5. Abueida, Clark, and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of $K_n - F$ for the graph-pairs of order 4 and 5, respectively, where F is a Hamiltonian cycle, a 1-factor, or a graph that is 1-regular except for one vertex of degree 0.

Furthermore, Shyu [24] investigated the problem of decomposing K_n into paths and stars with k edges, giving a necessary and sufficient condition for $k = 3$. In [23,25], Shyu considered the existence of a decomposition of K_n into paths and cycles with k edges, giving a necessary and sufficient condition for $k \in \{3, 4\}$. Shyu [27] investigated the problem of decomposing K_n into cycles and stars with k edges, settling the case $k = 4$. In [26], Shyu considered the existence of a decomposition of $K_{m,n}$ into paths and stars with k edges, giving a necessary and sufficient condition for $k = 3$.

Recently, Lee [16] and Lee and Lin [17] established necessary and sufficient conditions for the existence of (C_k, S_k)-decompositions of the complete bipartite graph and the complete bipartite graph with a 1-factor removed, respectively. In this paper, we consider the existence of a (C_k, S_k)-decomposition of the complete bipartite multigraph, giving necessary and sufficient conditions.

2. Preliminaries

We first collect some needed terminology and notation. Let G be a multigraph. For $S \subseteq V(G)$ and $T \subseteq E(G)$, we use $G[S]$ and $G-T$ to denote the subgraph of G induced by S and the subgraph of G obtained by deleting T, respectively. Furthermore, we use $\mu(xy)$ to denote the number of edges of G joining x and y, and $[v_1, \dots, v_k]$ to denote the k-cycle through vertices $v_1, \dots, v_k$ in order. The *degree* of a vertex x of G, denoted by $\deg_G(x)$, is the number of edges incident with x. The vertex of degree k in S_k is the *center* of S_k, and any vertex of degree 1 is a *leaf* of S_k. A *multistar* is a star with multiple edges allowed. Given an S_k-decomposition of G, a *central function* c from $V(G)$ to the set of nonnegative integers is defined as follows. For each $v \in V(G)$, $c(v)$ is the number of k-stars of the decomposition whose center is v. The following results are essential to our proof.

Proposition 2.1 (*Lin et al. [18]*). *A multistar H is S_k-decomposable if and only if there exists a nonnegative integer q such that $|E(H)| = qk$ and $\mu(wx) \le q$, where w is the center of H and x is any leaf.*

Proposition 2.2 (*Hoffman [15]*). *For a positive integer k, a multigraph G has an S_k-decomposition with central function c if and only if*

(1) $k \sum_{v \in V(G)} c(v) = |E(G)|$,

(2) *for all* $x, y \in V(G)$, $\quad \mu(xy) \le c(x) + c(y)$,

(3) *for all* $S \subseteq V(G)$, $\quad k \sum_{v \in S} c(v) \le \varepsilon(S) + \sum_{x \in S, y \in V(G)-S} \min\{c(x), \mu(xy)\}$,

where $\varepsilon(S)$ denotes the number of edges of G with both ends in S.

In the remainder of the paper, we use (A, B) to denote the bipartition of $\lambda K_{m,n}$, where $A = \{a_0, \dots, a_{m-1}\}$ and $B = \{b_0, \dots, b_{n-1}\}$.

3. S_k-decomposition of $\lambda K_{m,n}$

In this section necessary and sufficient conditions for the existence of an S_k-decomposition of $\lambda K_{m,n}$ are given. For the sake of simplicity, we use W to denote the vertex set of $\lambda K_{m,n}$.

Theorem 3.1. *For positive integers m and n with $m \ge n$, the complete bipartite multigraph $\lambda K_{m,n}$ is S_k-decomposable if and only if $m \ge k$ and*

$$\begin{cases} \lambda m \equiv 0 \pmod{k} & \text{if } n < k \\ \lambda mn \equiv 0 \pmod{k} & \text{if } n \ge k. \end{cases}$$

Proof (*Necessity*). Since the maximum size of a star in $\lambda K_{m,n}$ is m, the condition $k \le m$ is necessary. For $n < k$, all of the k-stars in a decomposition have centers in B. Since each vertex in B has degree λm and each k-star uses k edges, λm is divisible by k. For $n \ge k$, since $\lambda K_{m,n}$ has λmn edges and each subgraph in a decomposition has k edges, k must divide λmn.

(*Sufficiency*). Let $\lambda m = qk + r$, where q and r are integers with $0 \le r < k$. We first show that $q \ge \lambda$. If $r = 0$, then $\lambda m = qk$. Since $m \ge k$, we have $q \ge \lambda$. If $r > 0$, then $m > k$. Therefore, $q = \lambda m/k - r/k > \lambda - 1$. Since q is an integer, $q \ge \lambda$. For $r = 0$, let $G_j = \lambda K_{m,n}[\{b_j\} \cup A]$ with $j \in \{0, \dots, n-1\}$. Note that G_j is a multistar with $|E(G_j)| = \lambda m = qk$ and $\lambda K_{m,n} = \bigcup_{j=0}^{n-1} G_j$. Since $q \ge \lambda$ and $\mu(b_j x) = \lambda$ for each $b_j \in B$ and $x \in A$, Proposition 2.1 implies that G_j is S_k-decomposable for each j. Therefore, we may henceforth assume $r > 0$.

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