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Note

Extremal aspects of the Erdős–Gallai–Tuza conjecture

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ABSTRACT

Erdős, Gallai, and Tuza posed the following problem: given an n-vertex graph G, let $\tau_1(G)$ denote the smallest size of a set of edges whose deletion makes G triangle-free, and let $\alpha_1(G)$ denote the largest size of a set of edges containing at most one edge from each triangle of G. Is it always the case that $\alpha_1(G) + \tau_1(G) \leq n^2/4$? We also consider a variant on this conjecture: if $\tau_B(G)$ is the smallest size of an edge set whose deletion makes G bipartite, does the stronger inequality $\alpha_1(G) + \tau_B(G) \leq n^2/4$ always hold?

By considering the structure of a minimal counterexample to each version of the conjecture, we obtain two main results. Our first result states that any minimum counterexample to the original Erdős–Gallai–Tuza Conjecture has "dense edge cuts", and in particular has minimum degree greater than $n/2$. This implies that the conjecture holds for all graphs if and only if it holds for all triangular graphs (graphs where every edge lies in a triangle). Our second result states that $\alpha_1(G) + \tau_B(G) \leq n^2/4$ whenever G has no induced subgraph isomorphic to K_4^-, the graph obtained from the complete graph K_4 by deleting an edge. Thus, the original conjecture also holds for such graphs.

1. Introduction

Given an n-vertex graph G, say that a set $A \subseteq E(G)$ is *triangle-independent* if it contains at most one edge from each triangle of G, and say that $X \subseteq E(G)$ is a *triangle edge cover* if $G \setminus X$ is triangle-free. Throughout this paper, $\alpha_1(G)$ denotes the maximum size of a triangle-independent set of edges in G, while $\tau_1(G)$ denotes the minimum size of a triangle edge cover in G.

Erdős [1] showed that every n-vertex graph G has a bipartite subgraph with at least $|E(G)|/2$ edges, which implies that $\tau_1(G) \leq |E(G)|/2 \leq n^2/4$. Similarly, if A is triangle-independent, then the subgraph of G with edge set A is clearly triangle-free; by Mantel's Theorem, this implies that $\alpha_1(G) \leq n^2/4$. The Erdős–Gallai–Tuza conjecture is a common generalization of these upper bounds.

Conjecture 1.1 (*Erdős–Gallai–Tuza [5]*). *For every n-vertex graph G, $\alpha_1(G) + \tau_1(G) \leq n^2/4$.*

The conjecture is sharp, if true: consider the graphs K_n and $K_{n/2,n/2}$, where n is even. We have $\alpha_1(K_n) = n/2$ and $\tau_1(K_n) = \binom{n}{2} - n^2/4$, while $\alpha_1(K_{n/2,n/2}) = n^2/4$ and $\tau_1(K_{n/2,n/2}) = 0$. In both cases, $\alpha_1(G) + \tau_1(G) = n^2/4$, but a different term dominates in each case. More generally, the conjecture is sharp for any graph of the form $K_{r_1,r_1} \vee \cdots \vee K_{r_t,r_t}$, a fact which follows from the characterization of such graphs in [7] as the graphs achieving equality in the bound $\alpha_1(G) \leq \frac{n^2}{2} - |E(G)|$.

The original paper of Erdős, Gallai, and Tuza [5] considered the conjecture only for *triangular graphs*, which are graphs such that every edge lies in a triangle. This version of the conjecture was also stated by Erdős as Problem 17.2 of [2]. Later

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formulations of the conjecture, such as [3,8], dropped the triangularity requirement, and instead stated the conjecture for general graphs; this discrepancy was pointed out by Grinberg on MathOverflow [6], who asked if the two formulations were really equivalent. Our results in this paper imply that the two forms of the conjecture are equivalent, settling Grinberg's question.

Throughout the paper, we use the term *minimal counterexample* to refer to a vertex-minimal counterexample, that is, a graph G such that the property in question holds for every proper induced subgraph of G. When $S \subseteq V(G)$, we write $\overline{S}$ for the set $V(G) - S$, and we write $[S, \overline{S}]$ for the *edge cut* between S and $\overline{S}$, that is, the set of all edges with one endpoint in S and the other endpoint in $\overline{S}$.

In Section 2, we prove that if G is a minimal counterexample to Conjecture 1.1, then for every nonempty proper vertex subset S, the edge cut $[S, \overline{S}]$ has more than $|S|\,(n-|S|)/2$ edges. A small refinement of the argument shows that $\delta(G) > n/2$ whenever G is a minimal counterexample. Thus, any minimal counterexample is a triangular graph, so if Conjecture 1.1 holds for triangular graphs, then no counterexample exists.

We then consider the following stronger variant on Conjecture 1.1, proposed by Lehel (see [2]) and independently proposed by the author [7]. Let $\tau_B(G)$ denote the smallest size of an edge set X such that $G - X$ is bipartite; note that $\tau_B(G) \geq \tau_1(G)$.

Conjecture 1.2. *For every n-vertex graph G, $\alpha_1(G) + \tau_B(G) \leq n^2/4$.*

A partial result [7] towards Conjecture 1.2, and thus towards Conjecture 1.1, states that $\alpha_1(G) + \tau_B(G) \leq 5n^2/16$ for every graph G. In Section 3, we study the properties of a minimal counterexample to Conjecture 1.2, obtaining a "dense cuts" result similar to that of Section 2 (but somewhat more complicated to state). This theorem implies that if G has no induced subgraph isomorphic to K_4^-, then $\alpha_1(G) + \tau_B(G) \leq n^2/4$. Although this class of graphs is highly constrained, it includes K_n and $K_{n/2,n/2}$, the two extremes of the family of motivating sharpness examples.

2. Dense cuts in a minimal counterexample

Erdős, Gallai, and Tuza [5] showed that $\alpha_1(G) + \tau_1(G) \leq |E(G)|$ for all G, via the following argument: if $A \subseteq E(G)$ is triangle-independent, then $E(G) - A$ contains at least 2 edges from each triangle of G, so $E(G) - A$ is a triangle edge cover. This argument is "global", dealing with all edges in G; we "localize" it, dealing only with edges in some edge cut $[S, \overline{S}]$ for $S \subseteq V(G)$.

To avoid clutter, we write $f_1(G)$ for the sum $\alpha_1(G) + \tau_1(G)$.

Lemma 2.1. *If S is a nonempty proper subset of $V(G)$, then*

$$f_1(G) \leq f_1(G[S]) + f_1(G[\overline{S}]) + \left|[S, \overline{S}]\right|.$$

Proof. Let $A \subseteq E(G)$ be a largest triangle-independent set in G, let $G_1 = G[S]$, and let $G_2 = G[\overline{S}]$. For $i \in \{1, 2\}$, let $A_i = A \cap E(G_i)$, so that A_i is a triangle-independent set in G_i, and let $B = A \cap [S, \overline{S}]$. Since $|A_i|$ is a lower bound on $\alpha_1(G_i)$, we have

$$\alpha_1(G) = |A| = |A_1| + |A_2| + |B| \leq \alpha_1(G_1) + \alpha_1(G_2) + |B|.$$

Next, let X_i be a minimum triangle edge cover in G_i for $i \in \{1, 2\}$, so that $|X_i| = \tau_1(G_i)$, and let $Y = [S, \overline{S}] - B$. We claim that $X_1 \cup X_2 \cup Y$ is a triangle edge cover in G. Clearly X_i covers all triangles contained in $V(G_i)$, so it suffices to show that Y covers all triangles intersecting both S and $\overline{S}$. If T is such a triangle, then two edges of T lie in $[S, \overline{S}]$. Since $B \subseteq A$ and A is triangle-independent, at most one of these edges is contained in B; the other lies in Y. Hence $X_1 \cup X_2 \cup Y$ is a triangle edge cover in G, and we conclude that

$$\tau_1(G) \leq |X_1| + |X_2| + |Y| = \tau_1(G_1) + \tau_1(G_2) + \left(\left|[S, \overline{S}]\right| - |B|\right).$$

Combining the bounds on $\alpha_1(G)$ and $\tau_1(G)$ yields the desired inequality. □

Theorem 2.2. *Let G be a minimal counterexample to Conjecture 1.1. If S is a nonempty proper subset of $V(G)$, then $\left|[S, \overline{S}]\right| > \frac{1}{2}|S|\,(n-|S|)$, where $n = |V(G)|$.*

Proof. Let $G_1 = G[S]$ and let $G_2 = G[\overline{S}]$. Since G is a minimal counterexample, we have

$$\alpha_1(G_1) + \tau_1(G_1) \leq |S|^2/4,$$
$$\alpha_1(G_2) + \tau_1(G_2) \leq (n-|S|)^2/4.$$

By Lemma 2.1, it follows that

$$\alpha_1(G) + \tau_1(G) \leq \frac{n^2}{4} - \frac{|S|\,(n-|S|)}{2} + \left|[S, \overline{S}]\right|.$$

Since $\alpha_1(G) + \tau_1(G) > n^2/4$, the claim follows. □