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Large sets of wrapped Hamilton cycle decompositions of complete tripartite graphs

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ABSTRACT

Using the Katona–Kierstead definition of a Hamilton cycle in a uniform hypergraph, we settle the existence of wrapped Hamilton cycle decompositions (WHDs) of the λ-fold complete tripartite graph $\lambda K_{n,n,n}$ with one possible exception. The existence of large sets of WHDs of $\lambda K_{n,n,n}$ is also settled for all $n \equiv 0$, 1 or 3 (mod 4).

We also investigate the existence of wrapped Hamilton cycle decompositions of the λ-fold complete 3-uniform tripartite hypergraph $\lambda K^{(3)}_{n,n,n}$ which have the additional property that they can themselves be partitioned into WHDs (a result reminiscent of partitioning Steiner Quadruple Systems into BIBDs with block size 4).

1. Introduction

A *decomposition* of a graph $\mathcal{G} = (V, E)$ is a partition of the edge set E. A *Hamilton cycle decomposition* of $\mathcal{G}$ is a decomposition in which each element of the partition induces a Hamilton cycle. A Hamilton cycle decomposition of $\mathcal{G}$ is said to be *simple*, if it contains no repeated Hamilton cycles. The problem of constructing Hamilton cycle decompositions is a long-standing and well-studied one in combinatorial theory. Finding a Hamilton cycle decomposition of the complete graph K_n was solved in the 1890s by Walecki (see Lucas [13] or the recent articles by Alspach [1] and Bryant [4] for details). Walecki showed that K_n has a Hamilton cycle decomposition if and only if n is odd, while if n is even K_n has a decomposition into Hamilton cycles and one 1-factor. Laskar and Auerbach [11] extended this result in 1974, proving that the complete p-partite graph $K_{n,\ldots,n}$ has a Hamilton cycle decomposition when $n(p-1)$ is even, and a decomposition into Hamilton cycles and one 1-factor when $n(p-1)$ is odd.

As with many problems in design theory, it is natural to attempt a generalization to hypergraphs.

Definition 1.1. A *hypergraph* $\mathcal{H} = (V, E)$ consists of a finite set V of vertices with a family E of subsets of V, called *hyperedges* (or simply edges). $|V|$ is called the *order* of $\mathcal{H}$. If each (hyper)edge has size k, we say that $\mathcal{H}$ is a k-uniform hypergraph. In particular, the complete k-uniform hypergraph $K^{(k)}_n$ on a set V of n vertices is the hypergraph in which all k-subsets of V are hyperedges. (So $K^{(2)}_n$ is the complete graph K_n.)

A hypergraph is said to be *simple*, if it contains no repeated hyperedges. For a simple hypergraph $\mathcal{H}$, the λ-fold hypergraph $\lambda\mathcal{H}$ is formed on the same vertex set as $\mathcal{H}$ by defining $E(\lambda\mathcal{H})$ to contain λ copies of each hyperedge in $\mathcal{H}$.

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The notion of Hamiltonicity was generalized to k-uniform hypergraphs by Katona and Kierstead in their 1999 paper [8] as follows. Other definitions exist, such as the loose Hamilton cycle defined by Kühn and Osthus [10] and the Berge Hamilton cycle defined by Berge in his 1970 book [3], but these will not be considered in this paper. Let $Z_n = \{0, 1, \ldots, n-1\}$.

Definition 1.2 (*K–K Definition*)**.** Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of order n. A *K–K Hamilton cycle* $(v_0, v_1, \ldots, v_{n-1})_k$ in $\mathcal{H}$ is the hypergraph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\} = V$ and hyperedge set $\{\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} \mid i \in Z_n\}$ (calculating the subscripts modulo n).

Note that the K–K Hamilton cycle $H = (v_0, v_1, \ldots, v_{n-1})_k$ can also be represented by any cyclic shift of H or of $H^{-1} = (v_0, v_{n-1}, v_{n-2}, \ldots, v_1)_k$. Two such representatives of H are said to be *shift-equivalent*.

All Hamilton cycles are assumed to be K–K Hamilton cycles for the remainder of this paper. Meszka and Rosa [14], Baily and Stevens [2], and Xu and Wang [16], have investigated the existence of Hamilton cycle decompositions of the complete k-uniform hypergraph $K_n^{(k)}$ for various values of n and k. For the purposes of this paper, we will use the following definition.

Definition 1.3. $\mathcal{H}$ is a *k-uniform p-partite hypergraph* if (i) $|e| = k$ for each $e \in E(\mathcal{H})$, (ii) $V(\mathcal{H})$ is partitioned by $\{V_1, \ldots, V_p\}$, and (iii) $|e \cap V_i| \le 1$ for all $e \in E(\mathcal{H})$ and $1 \le i \le p$ (so must satisfy $k \le p$). Such a hypergraph is said to be *complete* if $E(\mathcal{H})$ is the set of all hyperedges satisfying properties (i)–(iii). The complete k-uniform p-partite hypergraph in which $|V_i| = n$ for $1 \le i \le p$ is denoted by $K_{p\times n}^{(k)} = K_{n,\ldots,n}^{(k)}$. (So $K_{n,\ldots,n}^{(2)}$ is the complete p-partite graph $K_{n,\ldots,n}$.)

In [9], Kuhl and Schroeder found the spectrum for Hamilton cycle decompositions of $K_{p\times n}^{(p)}$ for each prime p. In [15], M.W. Schroeder determined the spectrum for Hamilton cycle decompositions of $K_{r\times n}^{(r)}$ for any positive integer r.

Definition 1.4. For a simple hypergraph $\mathcal{H}$, a *large set* of Hamilton cycle decompositions of the hypergraph $\lambda\mathcal{H}$ is a partition of the set of all Hamilton cycles in $\mathcal{H}$ into pairwise disjoint Hamilton cycle decompositions of $\lambda\mathcal{H}$. Each Hamilton cycle decomposition in the large set is called a *small set*.

From the definition it is clear that in each large set of Hamilton cycle decompositions of $\lambda\mathcal{H}$, each small set is simple.

In [5,17], necessary and sufficient conditions for the existence of large sets of Hamilton cycle decompositions of λK_n have been given. Kang and the first author have determined the spectrum for large sets of Hamilton cycle decompositions of $\lambda K_{n,n}$ in [7,18]. The authors of this paper found the spectrum for large sets of Hamilton cycle decompositions of the λ-fold complete 3-uniform tripartite hypergraph $\lambda K_{n,n,n}^{(3)}$ in [19].

Of course, Definition 1.4 depends heavily on the definition of a Hamilton cycle when hypergraphs are considered. Here we focus on decompositions of $K_{n,n,n}^{(k)}$ with wrapped Hamilton cycles, defined as follows.

For convenience, we use $Z_n \cup \overline{Z}_n \cup \overline{\overline{Z}}_n$ to denoted the vertex set of $K_{n,n,n}^{(k)}$ with $k \in \{2, 3\}$, where

$$Z_n = \{0, 1, \ldots, n-1\}, \qquad \overline{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}, \quad \text{and} \quad \overline{\overline{Z}}_n = \{\overline{\overline{0}}, \overline{\overline{1}}, \ldots, \overline{\overline{n-1}}\}.$$

For each $a, b \in Z_n$, naturally define $\overline{a} + \overline{b} = \overline{a+b}$ and $\overline{\overline{a}} + \overline{\overline{b}} = \overline{\overline{a+b}}$.

For each $k \in \{2, 3\}$, a Hamilton cycle H in $K_{n,n,n}^{(k)}$ is said to be *wrapped* if it is of the form

$$H = (a_0, \overline{b}_0, \overline{\overline{c}}_0, a_1, \overline{b}_1, \overline{\overline{c}}_1, \ldots, a_{n-1}, \overline{b}_{n-1}, \overline{\overline{c}}_{n-1}), \tag{1}$$

where $\{a_0, a_1, \ldots, a_{n-1}\} = \{b_0, b_1, \ldots, b_{n-1}\} = \{c_0, c_1, \ldots, c_{n-1}\} = Z_n$ (being a K–K Hamilton cycle, each consecutive set of k vertices is an edge).

A *wrapped Hamilton cycle decomposition* of $\lambda K_{n,n,n}^{(k)}$, $\text{WHD}_\lambda^{(k)}(n)$, is a partition of the edge set of $\lambda K_{n,n,n}^{(k)}$ into sets, each of which induces a wrapped Hamilton cycle. It will cause no confusion if λ is omitted when $\lambda = 1$ and if (k) is omitted when $k = 2$. Clearly each $\text{WHD}_\lambda(n)$ contains $\frac{3\lambda n^2}{3n} = \lambda n$ wrapped Hamilton cycles and each $\text{WHD}_\lambda^{(3)}(n)$ contains $\frac{\lambda n^3}{3n} = \frac{\lambda n^2}{3}$ wrapped Hamilton cycles.

In [12], Lindner and Rodger introduced the concept of a tricycle system. A 3 m-*tricycle* of $K_{n,n,n}$, is a cycle of length 3 m of the form denoted by any cyclic shift of $(a_0, \overline{b}_0, \overline{\overline{c}}_0, a_1, \overline{b}_1, \overline{\overline{c}}_1, \ldots, a_{m-1}, \overline{b}_{m-1}, \overline{\overline{c}}_{m-1})$ or $(\overline{b}_0, a_0, \overline{\overline{c}}_{m-1}, \overline{b}_{m-1}, a_{m-1}, \ldots, \overline{\overline{c}}_0)$. The definition implies that the m elements in each of the sets $\{a_0, a_1, \ldots, a_{m-1}\}$, $\{b_0, b_1, \ldots, b_{m-1}\}$ and $\{c_0, c_1, \ldots, c_{m-1}\}$ are distinct. We can see that a $3n$-tricycle of $K_{n,n,n}$ is equivalent to a wrapped Hamilton cycle in $K_{n,n,n}$.

A 3 m-*tricycle system* of order n is an ordered pair $(Z_n \cup \overline{Z}_n \cup \overline{\overline{Z}}_n, T)$, where T is a collection of edge-disjoint 3 m-tricycles whose edges partition the edge set of $K_{n,n,n}$ based on the vertex set $Z_n \cup \overline{Z}_n \cup \overline{\overline{Z}}_n$. It is well known that a 3-tricycle system of order n is equivalent to a latin square of order n (see next section). Beyond this, not a lot is known. A $3n$-tricycle system of order n is actually a $\text{WHD}(n)$.

The main aim of this paper is to study wrapped Hamilton cycle decompositions of $\lambda K_{n,n,n}$ and large sets of wrapped Hamilton cycle decompositions of $\lambda K_{n,n,n}$.

2. Wrapped Hamilton cycle decompositions

In this section, we will prove that for any positive integer $n \ne 2, 6$, there exists a $\text{WHD}(n)$ and there exists a $\text{WHD}_3^{(3)}(n)$ which can be decomposed into n pairwise disjoint $\text{WHD}(n)$s.