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On extendability of Deza graphs with diameter 2

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ABSTRACT

A connected graph Γ of even order is *ℓ-extendable* if it is of order at least $2\ell + 2$, contains a matching of size ℓ, and if every such matching is contained in a perfect matching of Γ. A connected regular graph Γ is a *Deza graph*, if there exist integers λ and μ such that any two distinct vertices of Γ have either λ or μ common neighbours. In this paper we study extendability of regular graphs of even order and diameter 2. In particular, we show that every such graph is 1-extendable and we classify 2-extendable Deza graphs of even order and diameter 2.

1. Introductory remarks

Throughout this paper graphs are assumed to be finite and simple.

We first repeat the definition of ℓ-extendable graphs which were introduced in 1980 by Plummer [12]. A connected graph Γ of even order is *ℓ-extendable* if it is of order at least $2\ell + 2$, contains a matching of size ℓ, and if every such matching is contained in a perfect matching of Γ. Note that in his definition from 1980 Plummer did not require an ℓ-extendable graph to have order at least $2\ell + 2$. However, this additional assumption is convenient since it guarantees that ℓ-extendability implies $(\ell - 1)$-extendability. The family of ℓ-extendable graphs has been studied quite a lot in the last three decades (see for instance [2,3,9,11,13–17] and the references therein).

Extendability of the well known family of strongly regular graphs was also considered [8,10]. Recall that a connected graph Γ is *strongly regular* with parameters (n, k, λ, μ) if Γ is a k-regular graph on n vertices such that any two adjacent (nonadjacent respectively) vertices have exactly λ (μ respectively) common neighbours. It was proved in [8] that each strongly regular graph of even order $n \geq 4$ is 1-extendable. Moreover, the results of [8,10] show that the Petersen graph, the complete graph K_4 and the complete tripartite graph $K_{2,2,2}$ are the only strongly regular graphs of even order and valence at least 3 which are not 2-extendable.

The diameter of connected strongly regular graphs is of course 2 (except for the complete graphs). It thus seems natural to consider extendability of other regular graphs of diameter 2. In this paper we focus on the generalization of strongly regular graphs, the so-called Deza graphs (introduced in [4,6]). A connected graph Γ is said to be a *Deza graph* with parameters (n, k, λ, μ) if Γ is a k-regular graph on n vertices such that any two distinct vertices of Γ have either λ or μ common neighbours. Note that the diameter of such a graph is at most 2 provided that $\lambda, \ \mu \geq 1$.

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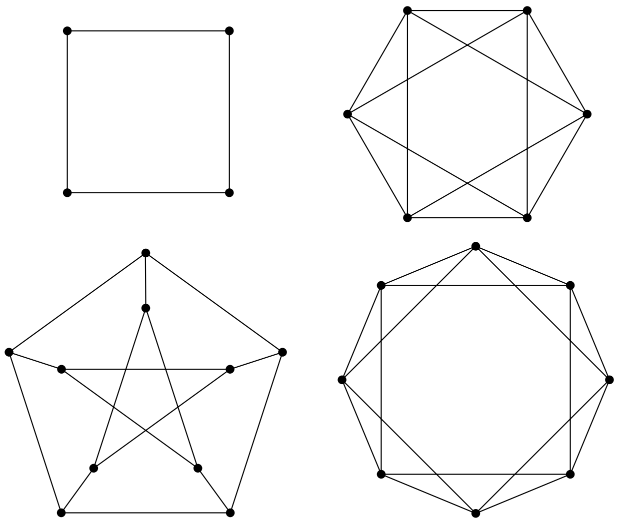

Fig. 1. The four non-2-extendable Deza graphs of even order and diameter 2.

In Section 2 we prove that every regular graph of even order and with diameter 2 is 1-extendable. In Section 3 we first determine the cubic graphs with diameter 2 that are not 2-extendable and classify all 2-extendable Deza graphs with valence $k \geq 4$, even order and diameter 2.

Theorem 1.1. *Let Γ be a Deza graph of even order and diameter 2. Then Γ is not 2-extendable if and only if it is isomorphic to one of the four graphs from Fig. 1.*

The four graphs from Fig. 1 are the 4-cycle C_4, the Petersen graph, the complete tripartite graph $K_{2,2,2}$, and the Cayley graph $\mathrm{Cay}(\mathbb{Z}_8; \{\pm 1, \pm 2\})$. The reason that the 4-cycle C_4 is included in this list is that its order is not large enough for it to be 2-extendable by our definition.

The general idea of the proof of Theorem 1.1 is similar to the one used in [10] for strongly regular graphs of even order. However, the proofs of the main steps for the more general class of Deza graphs are considerably more difficult. One of the key factors in the proof of Theorem 1.1 is the classical result of Tutte from 1947 giving a necessary and sufficient condition for a graph to contain a perfect matching. To state it we first need to fix some notation. For a graph Γ we denote its vertex set by $V = V(\Gamma)$ and for a vertex $v \in V$ we let $\Gamma(v)$ denote the set of neighbours of v. For a set $S \subseteq V$ we let $\Gamma - S$ be the subgraph of Γ induced on the set $V \setminus S$. Connected components of Γ will simply be called *components* of Γ. A component C of Γ is called *even* (*odd* respectively), if the cardinality of C is even (odd respectively). With $o(\Gamma)$ we denote the number of odd components of Γ.

We can now state the above mentioned result of Tutte.

Theorem 1.2 (*[5, Theorem 2.2.1]*). *A graph Γ has a perfect matching if and only if for every subset $S \subseteq V(\Gamma)$ we have $o(\Gamma - S) \leq |S|$.*

2. 1-extendability

In this section we prove that every regular graph of even order and diameter 2 is 1-extendable. Our proofs in this section will be along the lines of [8].

Lemma 2.1. *Let Γ be a regular graph of valence k, even order and diameter 2. Let $S \subseteq V$ be such that $\Gamma - S$ is not connected and let C be a connected component of $\Gamma - S$. Then there are at least k edges between C and S in Γ.*

Proof. Let $m = |C|$ and let $s = \min\{|\Gamma(v) \cap S|; v \in C\}$. We first claim that $s \geq 1$. Pick $v \in C$. Since $\Gamma - S$ is not connected, it has another connected component, say C'. Pick $v' \in C'$ and observe that the fact that the diameter of Γ is 2 implies that v and v' have a common neighbour, say u. Since clearly $u \in S$, this shows that $s \geq 1$.

Note that there are at least sm edges between C and S, and so it suffices to prove $sm \geq k$. Suppose to the contrary that $sm < k$. If $m \geq k - s + 1$, then $k > sm \geq s(k - s + 1)$. Rearranging terms we obtain

$$s^2 + k - ks - s = (s - k)(s - 1) > 0.$$

But this is impossible as $1 \leq s \leq k$. Therefore $m \leq k - s$. This shows that every vertex $v \in C$ has at most $k - s - 1$ neighbours in C. By regularity of Γ, v must have at least $s + 1$ neighbours in S, contradicting the definition of s. Therefore $sm \geq k$ and the result follows. ■