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Perfectly relating the domination, total domination, and paired domination numbers of a graph

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ABSTRACT

The domination number $\gamma(G)$, the total domination number $\gamma_t(G)$, the paired domination number $\gamma_p(G)$, the domatic number $d(G)$, and the total domatic number $d_t(G)$ of a graph G without isolated vertices are related by trivial inequalities $\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G) \leq 2\gamma(G)$ and $d_t(G) \leq d(G)$. Very little is known about the graphs that satisfy one of these inequalities with equality. We study classes of graphs defined by requiring equality in one of the above inequalities for all induced subgraphs that have no isolated vertices and whose domination number is not too small. Our results are characterizations of several such classes in terms of their minimal forbidden induced subgraphs. Furthermore, we prove some hardness results, which suggest that the extremal graphs for some of the above inequalities do not have a simple structure.

1. Introduction

Whenever two graph parameters, say ν and τ, are related by a simple inequality, say $\nu(G) \leq \tau(G)$ for every graph G, but the graphs G with $\nu(G) = \tau(G)$ do not have a simple structure, it makes sense to study the class of the so-called (ν, τ)-*perfect graphs*, the graphs for which every induced subgraph H satisfies $\nu(H) = \tau(H)$. Famous examples of this approach are the classical perfect graphs [3], where the two parameters are the clique number and the chromatic number, or the domination perfect graphs [22], where the two parameters are the domination number and the independent domination number. In the present paper we consider classes of (ν, τ)-perfect graphs, where ν and τ are well known domination parameters [9].

We consider finite, simple, and undirected graphs and use standard terminology and notation. Let D be a set of vertices of some graph G. The set D is a *dominating set of* G if every vertex of G that does not belong to D, has a neighbor in D. The set D is a *total dominating set of* G if it is dominating and the subgraph $G[D]$ of G induced by D does not have any isolated vertices. Finally, the set D is a *paired dominating set of* G if it is dominating and the graph $G[D]$ has a perfect matching. The *domination number* $\gamma(G)$ [9], the *total domination number* $\gamma_t(G)$ [5,14], and the *paired domination number* $\gamma_p(G)$ [10,16] of G are the minimum cardinalities of a dominating, a total dominating, and a paired dominating set of G, respectively.

These definitions immediately imply [1,9,10]

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G)$$

for every graph G for which $\gamma_p(G)$ is defined.

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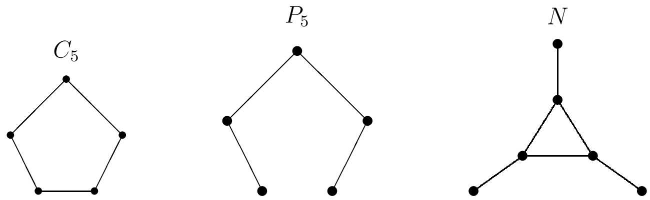

Fig. 1. The graphs C_5, P_5, and N.

For a set D of vertices of a graph G and a vertex u in D, let $P_G(D,u) = \{v \in V(G) \setminus D : N_G(v) \cap D = \{u\}\}$. The vertices in $P_G(D,u)$ are the *external private neighbors of* u (*in* G *with respect to* D). It is known that every graph G without isolated vertices has a minimum dominating set D such that, for every vertex u in D, the set $P_G(D,u)$ is non-empty [1]. Therefore, adding to such a dominating set D, for every vertex u in D, an external private neighbor of u, results in a paired dominating set of G, which implies

$$\gamma_p(G) \leq 2\gamma(G).$$

Only very little is known about the graphs that satisfy one of the above inequalities with equality. The trees T that satisfy $\gamma(T) = \gamma_t(T)$ [8], or $\gamma_t(T) = \gamma_p(T)$ [12,20], or $2\gamma(T) = \gamma_t(T)$ [11], or $2\gamma(T) = \gamma_p(T)$ [13] have been characterized constructively. Examples in [6] show that even very strong structural restrictions do not allow the strengthening of the inequality $\gamma_t(G) \leq 2\gamma(G)$. The relation between the total domination number and the paired domination number as well as paired dominating sets of a special structure were studied in [2,10,19,18].

The *domatic number* $d(G)$ of a graph G [5,21] is the maximum integer k such that the vertex set $V(G)$ of G can be partitioned into k dominating sets. The *total domatic number* $d_t(G)$ [4] and the *paired domatic number* $d_p(G)$ of G [10] are defined analogously. The definitions immediately imply [21]

$$d_p(G) \leq d_t(G) \leq d(G)$$

for every graph G without isolated vertices. Furthermore, since the union of at least two disjoint dominating sets is a total dominating set [21], we have

$$d_t(G) \geq \left\lfloor \frac{d(G)}{2} \right\rfloor$$

for every graph G without isolated vertices.

Having collected the relevant inequalities, we now consider some reasonable classes of "perfect" graphs. Since the total domination number and the paired domination number are only defined for graphs without isolated vertices, it does not make sense to require equalities involving these parameters for all induced subgraphs but only for those that have no isolated vertices. Therefore, if ν and τ are graph parameters with $\nu(G) \leq \tau(G)$ for every graph G, then let

$$\mathcal{G}(\nu,\tau) = \{G : \forall H \subseteq_{\text{ind}} G : \delta(H) \geq 1 \Rightarrow \nu(H) = \tau(H)\}$$

where $H \subseteq_{\text{ind}} G$ indicates that H is an induced subgraph of G.

Our first results are the following.

Note that all proofs are postponed to Section 2.

Theorem 1. *A graph G belongs to $\mathcal{G}(\gamma_t,\gamma_p)$ if and only if G is $\{C_5, P_5, N\}$-free.*

Theorem 2. *A graph G belongs to $\mathcal{G}(\gamma_t,2\gamma)$ if and only if G belongs to $\mathcal{G}(\gamma_p,2\gamma)$ if and only if G is $\{C_4, P_4\}$-free.*

See Fig. 1 for an illustration of the four special graphs in Theorem 1. In general, for a positive integer n, we denote the complete graph, the cycle, and the path of order n by K_n, C_n, and P_n, respectively.

Note that K_2 is a forbidden induced subgraph for graphs in $\mathcal{G}(\gamma,\gamma_t)$, $\mathcal{G}(d,d_t)$, and $\mathcal{G}(\gamma,\gamma_p)$. Therefore, these three classes of graphs consist exactly of all edgeless graphs. In view of this observation, it makes sense to restrict the considered induced subgraphs further. Therefore, for a positive integer k, we consider the class

$$\mathcal{G}_k(\nu,\tau) = \{G : \forall H \subseteq_{\text{ind}} G : (\delta(H) \geq 1 \text{ and } \gamma(H) \geq k) \Rightarrow \nu(H) = \tau(H)\}.$$

Note that $\mathcal{G}(\nu,\tau)$ coincides with $\mathcal{G}_1(\nu,\tau)$.

For $k = 2$, we obtain the following, where $2K_2$ denotes the disjoint union of two copies of K_2.

Theorem 3. *A graph G belongs to $\mathcal{G}_2(\gamma,\gamma_t)$ if and only if G is $\{C_5, 2K_2\}$-free.*

Theorem 4. *A graph G belongs to $\mathcal{G}_2(d,d_t)$ if and only if G is $\{P_4, 2K_2, G_1, G_2\}$-free.*

Theorem 5. *A graph G belongs to $\mathcal{G}_2(\gamma,\gamma_p)$ if and only if G is $\{C_5, 2K_2, N\}$-free.*

See Fig. 2 for an illustration of the graphs G_1 and G_2 in Theorem 4.