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Lower bounds on the number of edges in edge-chromatic-critical graphs with fixed maximum degrees

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ABSTRACT

In this article, we provide new lower bounds for the size of edge chromatic critical graphs with maximum degrees of 10, 11, 12, 13, 14, furthermore we characterize their class one properties.

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1. Introduction

Let V and E be the vertex set and edge set of a graph G, while $|V|$ and $|E|$ represent the cardinality of V and E of G, respectively. For a vertex x, set $N(x) = \{v : xv \in E(G)\}$ and $d(x) = |N(x)|$, the degree of x in G. We use Δ and δ to denote the maximum and the minimum degrees of G, respectively. For a vertex set S of G, set $N(S) = \cup_{x\in S} N(x)$. A *$k$-edge-coloring* of a graph G is a function $\phi : E(G) \mapsto \{1, \ldots, k\}$ such that any two adjacent edges receive different colors. The *edge chromatic number*, denoted by $\chi_e(G)$, of a graph G is the smallest integer k such that G has a k-edge-coloring. Vizing's Theorem [13] states that the edge chromatic number of a simple graph G is either Δ or $\Delta + 1$. A graph G is *class one* if $\chi_e(G) = \Delta$ and is *class two* otherwise. A class two graph G is *critical* if $\chi_e(G - e) < \chi_e(G)$ for each edge e of G. A critical graph G is *Δ-critical* if it has maximum degree Δ.

The following conjecture was proposed by Vizing [13] concerning the sizes of critical graphs.

Conjecture 1.1. *If $G = (V, E)$ is a critical simple graph, then $|E| \geq \frac{1}{2}(|V|(\Delta - 1) + 3)$.*

Some best known lower bounds of size of critical graphs are listed below [7,5,16,15,10]. Let G be a Δ-critical graph with average degree q, where $q = \frac{\sum_{v\in V(G)} d(v)}{|V|}$.

$$\begin{array}{llllll} \text{If } \Delta = 7, & q \geq 6. & \text{If } \Delta = 8, & q \geq \dfrac{20}{3}. & \text{If } \Delta = 9, & q \geq 7.3. \\ \text{If } \Delta = 10, & q \geq 8. & \text{If } \Delta = 11, & q \geq 8.6. & \text{If } \Delta = 12, & q \geq 9.25. \\ \text{If } 8 \leq \Delta \leq 17, & q \geq \dfrac{4}{7}(\Delta + 3). & \text{If } \Delta \geq 8, & q \geq \dfrac{2}{3}(\Delta + 1). & & \end{array}$$

We improve some of the earlier results in the following theorem: main theorem.

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Theorem 1.2. *Let G be a Δ-critical graph with $\Delta \geq 8$. Then $|E(G)| \geq \frac{|V(G)|}{2}q$ where $q = 8.25, 9, \frac{126}{13}, \frac{134}{13}, \frac{142}{13}$ for $\Delta = 10, 11, 12, 13, 14$ respectively.*

We show some lemmas in Section 2, and then provide our proof of the main theorem in Section 3.

2. Adjacency lemmas

Throughout this paper, G is a Δ-critical graph with $\Delta \geq 10$. A k-vertex (or, $(\leq k)$-vertex, $(\geq k)$-vertex) is a vertex of degree k (or $\leq k$, $\geq k$, respectively). A vertex w is a *k-neighbor* of x if $w \in N(x)$ and $d(w) = k$. Let V_k (or $V_{\leq k}$) be the set of vertices with degree k (or $\leq k$). Let $d_{\leq k}(x)$ denote the number of $(\leq k)$-vertices adjacent to x. Similarly define $d_{\geq k}(x)$. Let ϕ be the Δ-edge coloring of $G - xw$, $\phi(v)$ be the set of colors of the edges adjacent to the vertex v under edge coloring ϕ. A vertex v sees color j if v is adjacent to an edge colored by j. Denote by $P_{j,k}(v)_\phi$ the (j, k)-bi-colored path starting at v under edge coloring ϕ, or by $P_{j,k}(v)$ if there is no confusion. The following one belongs to Vizing [14], which will be abbreviated as VAL in this article.

VAL: If xw is an edge of a Δ-critical graph G, then x has at least $(\Delta - d(w) + 1)\Delta$-neighbors. Any vertex of G has at least two Δ-neighbors.

Adjacency Condition [17,11]: Let G be Δ-critical, $xw \in E(G)$ and $d(x) + d(w) = \Delta + 2$. The following hold: (1) every vertex of $N(x, w) \setminus \{x, w\}$ is a Δ-vertex; (2) every vertex of $N(N(x, w)) \setminus \{x, w\}$ is of degree at least $\Delta - 1$; and (3) if $d(x), d(w) < \Delta$, then every vertex of $N(N(x, w)) \setminus \{x, w\}$ is a Δ-vertex.

Through this paper, without loss of generality, under coloring ϕ, edges incident with x in $G - xw$ are colored by $1, 2, \ldots, d - 1$, while those incident with w are colored by $\Delta - k + 2, \ldots \Delta$ where $d = d(x)$, $k = d(w)$.

Let C_1 be the set of colors present at only one of x, w and C_2 be the set of colors present at both. Further let C_{11} be the set of colors present only at x, and C_{12} be the set of colors present only at w. We may assume that $C_1 = C_{11} \cup C_{12} = \{1, \ldots, \Delta - k + 1\} \cup \{d, d + 1, \ldots, \Delta\}$ and $C_2 = \{\Delta - k + 2, \ldots, d - 1\}$, where $C_2 = \emptyset$ if $d + k = \Delta + 2$. $|C_1| = 2\Delta - d - k + 2$, $|C_2| = d + k - \Delta - 2$. Let $C_v = \{i : \text{ vertex } v \text{ misses color } i\}$.

Lemma 2.1 ([8]). *Let xw be an edge of G with $d(x) + d(w) = \Delta + 2$ and $d(x), d(w) < \Delta$. Then every vertex of $N(N(N(x, w))) \setminus \{x, w, N(x, w), N(N(x, w))\}$ (assume that it is not empty) is adjacent to all Δ-vertices.*

In order to give improved adjacency properties on the i-vertex, we provide some claims. First two claims are equivalent to Facts 1 and 2 in [9], and for the purpose of convenience of uniform discussion, we re-write them as Claims A and B.

Claim A. *For each neighbor w_j of w in $G - xw$ where $\phi(ww_j) = j$ present only at w, then w_j must see each color in C_1.*

Claim A will be often used in the discussion through this paper without notifying.

Claim B. *For each neighbor x^i of x where $\phi(xx^i) = i$ present only at x, then x^i must see each color in C_1. Note that x has at least $\Delta - k + 1$ such x^i.*

Due to Claim B, we call a swapping (i, j) a *nice* swapping if it does not change the *set* of colors of edges incident with x and w in $G - xw$.

Claim C. *For a neighbor w_b of w where $b \in C_2$, if one of such w_b's misses a color in C_1, then we could assume that one of those w_b's misses color 1. Note that we can only assure there is one such vertex w_b.*

We assume, without loss of generality, that w_b misses Δ but sees 1, then we swap color 1 with the missing color along the path starting at w_b, by Claim B, this swapping is a *nice* one because it does not affect the colors of edges that are incident with x, w. So w_b misses color 1.

Claim D which follows is similar to Fact 4 in [9] but it is slightly stronger. So the proof is provided in the appendix.

Claim D. *Let x and w be adjacent in Δ-critical graph G with $d(x) = d$, $d(w) = k$. $G - xw$ has a Δ-edge coloring ϕ. Let $xx^a y$ be a path in $G - xw$ where $\phi(xx^a) = a \in C_{11}$ and $y \neq w$ such that $\phi(x^a y) \in C_1$. Then y must see each color in C_1, that is, $d(y) \geq 2\Delta - d - k + 2$. Note that there are $2\Delta - d - k + 1$ such y's, and some of them may be adjacent to vertices in $N(x)$.*

Lemma 2.2. *For a Δ-edge coloring ϕ of $G - xw$ with $d(x) = d$, $d(w) = k$ (see Fig. 1), let $xx^\alpha y$ and $xx^r u$ be paths that start at x, where $\phi(xx^\alpha) = \alpha$ present only at x and $\phi(xx^r) = r$ is a color in C_2. If there is a vertex $w_j \in N(w)$, where $\phi(ww_j) = j \in C_{12}$, and w_j misses $r \in C_2$, or if there is a $w_r \in N(w)$ with $\phi(ww_r) = r \in C_2$, and w_r misses a color in C_1, then we have the following:*

(i) x^α must see $r \in C_2$. (ii) y sees each color in C_1 and r; further, if $\phi(x^\alpha y) = r \in C_2$, then y sees each color in C_1 and color $r'(\neq r)$ if there is a $w_{j'} \in N(w)$ ($j' \in C_{12}$) missing $r' \in C_2$, or there is a $w_{r'} \in N(w)$ with $\phi(ww_r') = r'$ and $w_{r'}$ misses a color in C_1. (iii) x^r must see each color in C_1 and also color r' as described in (ii). *(iv) u sees each color in C_1 and also sees r' as described in* (ii).

Proof. The proof consists of two parts: Part I and Part II. Part I: If there is a vertex $w_j \in N(w)$, where $\phi(ww_j) = j \in C_{12}$, and w_j misses a color $r \in C_2$, then our results hold. Part II: If there is a $w_r \in N(w)$ with $\phi(ww_r) = r \in C_2$, and w_r misses a color in C_1, then our results hold.

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