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The surviving rate of digraphs☆

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ABSTRACT

Let $\overrightarrow{G}$ be a connected digraph with $n \geq 2$ vertices. Suppose that a fire breaks out at a vertex v of $\overrightarrow{G}$. A firefighter starts to protect vertices. At each time interval, the firefighter protects k vertices not yet on fire. Afterward, the fire spreads to all unprotected neighbors that are heads of some arcs starting from the vertices on fire. Let $\mathrm{sn}_k(v)$ denote the maximum number of vertices in $\overrightarrow{G}$ that the firefighter can save when a fire breaks out at vertex v. The k-surviving rate $\rho_k(\overrightarrow{G})$ of $\overrightarrow{G}$ is defined as $\sum_{v\in V(\overrightarrow{G})} \mathrm{sn}_k(v)/n^2$.

In this paper, we consider the k-surviving rate of digraphs. Main results are as follows: (1) if $\overrightarrow{G}$ is a k-degenerate digraph, then $\rho_k(\overrightarrow{G}) \geq \frac{1}{k+1}$; (2) if $\overrightarrow{G}$ is a planar digraph, then $\rho_2(\overrightarrow{G}) > \frac{1}{40}$; (3) if $\overrightarrow{G}$ is a planar digraph without 4-cycles, then $\rho_1(\overrightarrow{G}) > \frac{1}{51}$.

1. Introduction

In 1995, Hartnell [7] introduced the firefighter problem on a finite graph G. Assume that a fire breaks out at a vertex v of G. A firefighter (or defender) chooses a vertex not yet on fire to protect. Then the firefighter and the fire alternately move on the graph. Once a vertex has been chosen by the firefighter, it is considered protected or safe from any further moves of the fire. After the firefighter's move, the fire makes its move by spreading to all vertices which are adjacent to the vertices on fire, except for those that are protected. The process ends when the fire can no longer spread.

Let $\mathrm{sn}(v)$ denote the maximum number of vertices in G that the firefighter can save when a fire breaks out at vertex v. Determining for a graph G, vertex $v \in V(G)$ and an integer l, whether $\mathrm{sn}(v) \geq l$ is NP-complete, even when G is restricted to bipartite graphs [11], cubic graphs [8] and trees with maximum degree three [4]. For a survey of related results the reader is referred to [5].

The *surviving rate* $\rho(G)$ of a graph G with n vertices was introduced by Cai and Wang [2] and is defined to be the average proportion of vertices that can be saved when a fire breaks out at one vertex of the graph. More generally, for an integer $k \geq 1$, the k-firefighter problem is the same as the firefighter problem, except that at each move, the firefighter protects k vertices. We use $\mathrm{sn}_k(v)$ to denote the maximum number of vertices in G that the firefighter can save when a fire breaks out at vertex v. The *k-surviving rate* $\rho_k(G)$ of a graph G with n vertices is defined by

$$\rho_k(G) = \frac{\sum\limits_{v\in V(G)} \mathrm{sn}_k(v)}{n^2}.$$

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In particular, $\rho_1(G) = \rho(G)$. By the definition, it is evident that for any integer $k \geq 1$ and a graph G on n vertices, $0 \leq \rho_k(G) < 1$, and $\rho_k(G) = 0$ if and only if $n = 1$. Thus, we always assume that $n \geq 2$ in the following arguments.

Wang et al. [13] proved that for any $k \geq 1$, the k-surviving rate of almost all graphs is arbitrarily close to zero and therefore they began studying classes of special graphs, e.g., planar graphs, with the k-surviving rate bounded away from zero. In [3], Esperet et al. defined the firefighter number for a class of graph $\mathcal{C}$. Formally, the firefighter number for a class of graph $\mathcal{C}$, denoted by $ff(\mathcal{C})$, is the minimum integer k such that there exists $\epsilon > 0$ and an integer N so that every $G \in \mathcal{C}$ with at least N vertices has $\rho_k(G) > \epsilon$. The graph $K_{2,n}$ shows that for $\mathcal{P}$, the class of planar graphs, $ff(\mathcal{P}) \geq 2$. The firefighter number of the class of planar graphs with girth at least seven is one [14]. The firefighter number of planar graphs with girth five and six remains open. Two independent proofs have shown that $ff(\mathcal{P}) \leq 4$ [3,9], and this was recently improved as $ff(\mathcal{P}) \leq 3$[6,10] and it was conjectured that $ff(\mathcal{P}) = 2$ [3]. This conjecture was confirmed for planar graphs without 3-cycles [3], without 4-cycles [15], or without 6-cycles [12]. For other results on the surviving rate of graphs readers are referred to [1,16].

In this paper, we consider the firefighter problem on a digraph D. Suppose that a fire breaks out at a vertex v of D. A firefighter chooses a vertex not yet on fire to protect. Once a vertex has been chosen by the firefighter, it is considered protected or safe from any further moves of the fire. After the firefighter's move, the fire spreads to all unprotected neighbors that are heads of some arcs starting from the vertices on fire. The process ends when the fire can no longer spread. Similarly, we use $\mathrm{sn}_k(v)$ to denote the maximum number of vertices in D that the firefighter can save when a fire breaks out at vertex v. The *k-surviving rate* $\rho_k(D)$ of a digraph D with n vertices is defined by

$$\rho_k(D) = \frac{\sum\limits_{v \in V(D)} \mathrm{sn}_k(v)}{n^2}.$$

We first consider the k-surviving rate on a digraph D by showing that $\rho_k(D) \geq \frac{1}{k+1}$ for a k-degenerate digraph D. Then we consider a planar digraph D and show the following results: (1) $\rho_2(D) > \frac{1}{40}$; (2) $\rho(D) > \frac{1}{51}$ if D has no 4-cycles.

2. Notation

A *plane* graph is a particular drawing in the Euclidean plane of a certain planar graph. For a plane graph G, we denote its vertex set, edge set, and face set by $V(G)$, $E(G)$, $andF(G)$, respectively. Let $n = |V(G)|$. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \dots u_m]$ if $u_1, u_2, \dots, u_m$ are the vertices of $b(f)$ in the clockwise order. Repeated occurrences of a vertex are allowed. The degree of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. For $x \in V(G) \cup F(G)$, let $d_G(x)$, or simply $d(x)$, denote the degree of x in G. A face of degree k, at least k, or at most k is called a *k-face*, *k^+-face*, or *k^--face*, respectively.

A *digraph D* is an order pair (V, A) consisting of a set V of vertices and a set A, disjoint from V, of arcs, together with an incidence function ψ_D that associates with each arc of D an ordered pair of vertices of D. If a is an arc and $\psi_D(a) = (u, v)$, then a is said to join u to v; we also say that u dominates v. The vertex u is called the *tail* of a, and the vertex v its *head*; they are the two ends of a. The vertices which dominate a vertex v are its in-neighbors, those which are dominated by the vertex its out-neighbors. These sets are denoted by $N_D^-(v)$ and $N_D^+(v)$, respectively.

Given a graph G, we may obtain a digraph by replacing each edge by just one of the two possible arcs with the same ends. Such a digraph is called an *orientation* of G. We often use the symbol $\overrightarrow{G}$ to express an orientation of G. An orientation of a simple graph is referred to as an *oriented graph*. The degree of a vertex v in a digraph D is simply the degree of v in G, the underlying graph of D. The *indegree* $d_D^-(v)$ of a vertex v in D is the number of arcs with v as head, and the *outdegree* $d_D^+(v)$ of v is the number of arcs with v as a tail.

Let $k \geq 1$ be an integer. A class of graphs, $\mathcal{G}$, is said to be *k-good* if the k-surviving rate of any graph $G \in \mathcal{G}$ is greater than or equal to a positive constant c.

3. Degenerate digraphs

In this section, we consider the k-surviving rate on digraphs.

Theorem 1. *Let $0 < \varepsilon \leq 1$ be a real number and $k \geq 1$ be an integer. If D is a digraph with $n(\geq k+1)$ vertices and m arcs such that $m \leq (k+1-\varepsilon)n$, then $\rho_k(D) \geq \frac{\varepsilon}{k+1}$.*

Proof. Let V_{out}^* denote the set of vertices with outdegree at most k and $n^* = |V_{out}^*|$. Clearly, $\mathrm{sn}_k(v) = n - 1$ for any vertex $v \in V_{out}^*$. As $m \leq (k+1-\varepsilon)n$ and $m = \sum_{v \in V(D)} d^+(v)$, we have

$$(k+1-\varepsilon)n \geq \sum_{v \in V(D)} d^+(v) = \sum_{v \in V_{out}^*} d^+(v) + \sum_{v \in V(D) \setminus V_{out}^*} d^+(v) \geq (k+1)(n - n^*).$$