Discrete Mathematics 334 (2014) 26–37

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the size of 3-uniform linear hypergraphs

Niraj Khare [1]

The Ohio State University, Columbus, OH, USA

ARTICLE INFO

Article history:
Received 14 July 2013
Received in revised form 29 April 2014
Accepted 14 June 2014
Available online 11 July 2014

This article is dedicated to Prof. Ákos Seress for his contribution to Combinatorics, and inspiring love for the field in his students

Keywords:
Uniform hypergraphs
Linear hypergraphs
Matching
Maximum degree

ABSTRACT

This article provides bounds on the size of a 3-uniform linear hypergraph with restricted matching number and maximum degree. In particular, we show that if a 3-uniform, linear family $\mathcal{F}$ has maximum matching size ν and maximum degree Δ such that $\Delta \geq \frac{23}{6}\nu\left(1+\frac{1}{\nu-1}\right)$, then $|\mathcal{F}| \leq \Delta\nu$.

1. Introduction

Let V be a set of vertices and let $\mathcal{F} \subseteq 2^V$ be a set of distinct subsets of V. A set system $\mathcal{F}$ is k-uniform for a positive integer k if $|A| = k$ for all $A \in \mathcal{F}$. A set system $\mathcal{F}$ is linear if $|A \cap B| \leq 1$ for all distinct A, B in $\mathcal{F}$. For a hypergraph $\mathcal{G} = (V, \mathcal{F})$, the set V is called the set of vertices of $\mathcal{G}$ and the set $\mathcal{F} \subseteq 2^V$ is called the set of hyper-edges of $\mathcal{G}$. The size of a k-uniform linear hypergraph $\mathcal{G} = (V, \mathcal{F})$ is $|\mathcal{F}|$—the number of its hyper-edges. A matching in $\mathcal{G}$ (or $\mathcal{F}$) is a collection of pairwise disjoint hyper-edges of $\mathcal{G}$. The size of a maximum matching in $\mathcal{F}$ shall be denoted by $\nu(\mathcal{F})$. Also, degree of a vertex and maximum degree of $\mathcal{G}$ is defined in a usual familiar way. For any $x \in V$, define $\mathcal{F}_x = \{A \in \mathcal{F} \mid x \in A\}$ and $\Delta(\mathcal{F}) = \max\{|\mathcal{F}_x| \mid x \in V\}$. The objective of this article is to find a bound on the size of $\mathcal{F}$ for given values of $\Delta(\mathcal{F})$ and $\nu(\mathcal{F})$. Throughout the remainder of this article unless otherwise stated, $\mathcal{F}$ shall be a 3-uniform linear set system with maximum matching size $\nu(\mathcal{F}) = \nu$ and maximum degree $\Delta(\mathcal{F}) = \Delta$. Also, for any set system $\mathcal{H}$ and $\mathcal{B} \subseteq \mathcal{H}$, we shall denote by $X_{\mathcal{B}}$ the vertex set of $\mathcal{B}$ that is $X_{\mathcal{B}} := \bigcup_{A \in \mathcal{B}} A$.

The problem of bounding the size of a uniform family by restricting matching size and maximum degree has been studied for simple graphs in [4,2]. These articles were in turn inspired by the sunflower lemma due to Erdős and Rado (see [7]). A *sunflower* with s petals is a collection of sets $A_1, A_2, \ldots, A_s$ and a set X (possibly empty) such that $A_i \cap A_j = X$ whenever $i \neq j$. The set X is called the core of the sunflower. A linear family admits two kinds of sunflowers: (i) a matching is a sunflower with an empty core; (ii) a collection of hyper-edges incident at a vertex. It is a well-known result (due to Erdős–Rado [7]) that a k-uniform set system, with more members than $k!(s-1)^k$ admits a sunflower with s petals (for a proof see [1]). Other bounds that ensure the existence of a sunflower with s petals are known in the case of $s = 3$ with block size k (see [11]). However, not much progress has been made towards the general case. This article considers the dual problem of finding

E-mail addresses: khare.9@osu.edu, nirajkhare@math.ohio-state.edu.

[1] Current address: Texas A&M University, Qatar (Campus), Qatar.

http://dx.doi.org/10.1016/j.disc.2014.06.016
0012-365X/

the maximum size of a 3-uniform, linear family $\mathcal{F}$ that admits no sunflower with s petals, i.e., $s > \nu(\mathcal{F})$ and $s > \Delta(\mathcal{F})$. In particular, we find the maximum size of a 3-uniform, linear family $\mathcal{F}$ that admits no sunflower with $\nu + 1$ petals of empty core and no sunflower with $\Delta + 1$ petals of core cardinality one. Thus, this problem belongs to the class of Turán problems that find a bound on the size of the edge set of a graph (or a hypergraph) that avoids a substructure or substructures (see [3]). A significant recent result in this area is [8] where the aim is to find a bound on the size of a uniform family subject to its restricted matching size and number of vertices. This generalizes for hypergraphs a result on the size of the edge set of a simple graph due to Erdős and Gallai [6]. This article aims to share some new bounds and also brings forth some interesting questions in this well studied area. The following remark on the size of a family shall be useful later in proving the main result.

Remark 1. For a positive integer Δ, let a 3-uniform family $\mathcal{G}$ be a sunflower with Δ petals and core of size one. For any positive integer ν, let $\mathcal{F}$ consist of ν components where each component is isomorphic to $\mathcal{G}$. It is obvious that $\nu(\mathcal{F}) = \nu$, $\Delta(\mathcal{F}) = \Delta$ and $|\mathcal{F}| = \Delta\nu$.

The main result, Theorem 3, establishes sunflowers as maximal examples of 3-uniform, linear families $\mathcal{F}$ that have maximum number of hyper-edges for restricted values of maximum matching $\nu(\mathcal{F})$ and maximum degree $\Delta(\mathcal{F})$ if degree is approximately four times the matching size. It is natural to find an extension of the result for k-uniform linear families. The general result is not the focus of the article. However, if Δ is not large enough relative to ν then there are families such that $|\mathcal{F}| > \Delta(\mathcal{F})\nu(\mathcal{F})$. For example projective plane naturally induces a hypergraph $\mathcal{F}$ with uniformity $k = q + 1$, maximum degree $q + 1$ and matching number 1, while the number of edges $|\mathcal{F}| = q^2 + q + 1$.

2. Results

Our aim in this article is to prove the following two results.

Theorem 2. *Let $\mathcal{F}$ be a 3-uniform linear set system with maximum matching size $\nu(\mathcal{F}) = \nu$ and maximum degree $\Delta(\mathcal{F}) = \Delta$. If $\Delta \geq 5$, then $|\mathcal{F}| \leq 2\Delta\nu$.*

The main result, of this article is a tighter bound in the case Δ is approximately greater than 4ν. The precise statement follows.

Theorem 3 (*The Main Result*)**.** *Let $\mathcal{F}$ be a 3-uniform linear set system with maximum matching size $\nu(\mathcal{F}) = \nu$ and maximum degree $\Delta(\mathcal{F}) = \Delta$. If $\Delta \geq \frac{23}{6}\nu(1 + \frac{1}{\nu-1})$, then $|\mathcal{F}| \leq \Delta\nu$.*

Let ν be any positive integer. It is worthwhile to note that there are 3-uniform linear families $\mathcal{F}$ with $\nu = \nu(\mathcal{F})$ such that $|\mathcal{F}| > \Delta(\mathcal{F})\nu(\mathcal{F})$. In the next section, we construct such families and thus establish the importance of the main result-Theorem 3.

3. Families with large size

Let $\mathcal{F}$ be a 3-uniform linear family with $\Delta := \Delta(\mathcal{F})$ and $\nu := \nu(\mathcal{F})$. We present some examples such that $|\mathcal{F}| > \Delta\nu$.

(i) There are block designs $\mathcal{F}$ with block size three such that $|\mathcal{F}| \geq \nu(\mathcal{F})\Delta(\mathcal{F})$. For example, consider Steiner triples $S(n, 3, 2)$. A Steiner system $S(n, k, r)$ is a set system on n vertices such that each member has cardinality k and every r-subset of vertices is contained in a unique member (also called block) of the family $S(n, k, r)$. It is well known that $S(n, 3, 2)$ exists if and only if $n \geq 3$, and $n \equiv 1 (\mathrm{mod}\ 6)$ or $n \equiv 3 (\mathrm{mod}\ 6)$ (see [5], for instance).
   - If $n = 6m + 1$ and $\mathcal{F}$ is an $S(n, 3, 2)$ then $|\mathcal{F}| = \frac{1}{3}\binom{6m+1}{2} = m(6m + 1)$, $\Delta(\mathcal{F}) = 3m$, and $\nu(\mathcal{F}) \leq 2m$, so $|\mathcal{F}| > \Delta(\mathcal{F})\nu(\mathcal{F})$.

(ii) By the method given in [2], we can construct a simple graph G for any $\Delta := \Delta(G)$ and $\nu := \nu(G)$ such that $|E(G)| = \nu\Delta + \lfloor \frac{\nu}{\lceil \frac{\Delta}{2} \rceil} \rfloor \lfloor \frac{\Delta}{2} \rfloor$. Note that if $2 \leq \Delta \leq 2\nu$ then $|E(G)| > \Delta\nu$. Let Y be a set such that $Y \cap V(G) = \emptyset$ and $|Y| = |E(G)|$. We order the edges $\{e_1, e_2, \ldots, e_{|E(G)|}\}$ in $E(G)$ randomly and let $Y = \{y_1, y_2, \ldots, y_{|E(G)|}\}$. We define a linear, 3-uniform family $\mathcal{F}$ such that $\nu(\mathcal{F}) = \nu(G)$ and $\Delta(\mathcal{F}) = \Delta(G)$. For $i \in \{1, 2, \ldots, |E(G)|\}$, let $A_i := e_i \cup \{y_i\}$. Now let $\mathcal{F} := \{A_i | \ i \in \{1, 2, \ldots, |E(G)|\}\}$. It is obvious that $\mathcal{F}$ is a 3-uniform, linear family. Also note that $\nu(\mathcal{F}) = \nu$, $\Delta(\mathcal{F}) = \Delta$ and $|\mathcal{F}| = |E(G)|$. Thus, $|\mathcal{F}| = |E(G)| = \nu\Delta + \lfloor \frac{\nu}{\lceil \frac{\Delta}{2} \rceil} \rfloor \lfloor \frac{\Delta}{2} \rfloor > \Delta\nu$.

Theorem 3 states that if Δ is large enough compared to ν then $|\mathcal{F}| \leq \nu\Delta$. On the other hand the example in part (ii) above shows that for any positive integer ν, there are families $\mathcal{F}$ such that $|\mathcal{F}| > \Delta\nu$ with $2 \leq \Delta \leq 2\nu$. It would be interesting to determine the exact value $f(\nu)$ so that for any 3-uniform, linear family $\mathcal{F}$ with $\Delta(\mathcal{F}) = \Delta \geq f(\nu)$ and $\nu(\mathcal{F}) = \nu$, we have $|\mathcal{F}| \leq \nu\Delta$.