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Light C_4 and C_5 in 3-polytopes with minimum degree 5

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ABSTRACT

Let $w_P(C_l)$ ($w_T(C_l)$) be the minimum integer k with the property that every 3-polytope (respectively, every plane triangulation) with minimum degree 5 has an l-cycle with weight, defined as the degree-sum of all vertices, at most k.

In 1998, O.V. Borodin and D.R. Woodall proved $w_T(C_4) = 25$ and $w_T(C_5) = 30$. We prove that $w_P(C_4) = 26$ and $w_P(C_5) = 30$.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a plane graph G is the number of its incident edges. A *k-vertex* (*k-neighbor*, *k-face*) is a vertex (neighbor, face) with degree k, a k^+*-vertex* has degree at least k, etc. The minimum vertex degree of G is $\delta(G)$. We will drop the arguments whenever this does not lead to confusion.

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but $d(x) \geq 3$ for every vertex and face x. As proved by Steinitz [27], the 3-connected plane graphs are planar representations of the convex 3-dimensional polytopes, called hereafter 3-polytopes.

In this note, we consider the class $\mathbf{M_5}$ of NPMs with $\delta = 5$ and its subclasses $\mathbf{P_5}$ of 3-polytopes and $\mathbf{T_5}$ of plane triangulations, where we define a triangulation to be simple (without loops or multiple edges), so that $\mathbf{T_5} \subset \mathbf{P_5} \subset \mathbf{M_5}$. A cycle on k vertices is denoted by C_k, and S_k stands for a k-star centered at a 5-vertex.

In 1904, Wernicke [28] proved that if $M_5 \in \mathbf{M_5}$ then M_5 contains a vertex of degree 5 adjacent to a vertex of degree at most 6. This result was strengthened by Franklin [15] in 1922 to the existence of a vertex of degree 5 with two neighbors of degree at most 6. In 1940, Lebesgue [22, p. 36] gave an approximate description of the neighborhoods of vertices of degree 5 in a $T_5 \in \mathbf{T_5}$.

Given a graph H, the *weight* $w_M(H)$ is the maximum over $M_5 \in \mathbf{M_5}$ of the minimum degree-sum of the vertices of H over subgraphs H of M_5. The *weights* $w_P(H)$ and $w_T(H)$ are defined similarly for $\mathbf{P_5}$ and $\mathbf{T_5}$, respectively.

The bounds $w_M(S_1) \leq 11$ (Wernicke [28]) and $w_M(S_2) \leq 17$ (Franklin [15]) are tight. It was proved by Lebesgue [22] that $w_M(S_3) \leq 24$ and $w_M(S_4) \leq 31$, which were improved much later to the following tight bounds: $w_M(S_3) \leq 23$ (Jendrol'–Madaras [17]) and $w_M(S_4) \leq 30$ (Borodin–Woodall [9]). Note that $w_M(S_3) \leq 23$ easily implies $w_M(S_2) \leq 17$ and immediately follows from $w_M(S_4) \leq 30$ (it suffices to delete a vertex of maximum degree from a star of the minimum weight).

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It follows from Lebesgue [22, p. 36] that $w_T(C_3) \leq 18$. In 1963, Kotzig [21] gave another proof of this fact and conjectured that $w_T(C_3) \leq 17$; the bound 17 is easily shown to be tight.

In 1989, Kotzig's conjecture was confirmed by Borodin [1] in a more general form, by proving $w_M(C_3) = 17$. Another consequence of this result is confirming a conjecture of Grünbaum [16] of 1975 that for every 5-connected planar graph the cyclic connectivity (defined as the minimum number of edges to be deleted to obtain two components each containing a cycle) is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [26]).

It also follows from Lebesgue [22, p. 36] that $w_T(C_4) \leq 26$ and $w_T(C_5) \leq 31$. In 1998, Borodin and Woodall [9] proved the following.

Theorem 1 (*Borodin–Woodall [9]*). *For the class of plane triangulations with minimum degree* 5, $w_T(C_4) = 25$ *and* $w_T(C_5) = 30$.

The *height* of a subgraph H of graph G is the maximum degree of vertices of H in G. Now let $\varphi_M(H)$ ($\varphi_P(H)$, $\varphi_T(H)$) be the minimum integer k with the property that every normal plane map (3-polytope, plane triangulation) with minimum degree 5 has a copy of H with all vertices of degree at most k.

It follows from Franklin [15] that $\varphi_M(S_2) = 6$. From $w_M(C_3) = 17$ (Borodin [1]), together with a simple example proving $\varphi_M(C_3) \geq 7$, we have $\varphi_M(C_3) = 7$. In 1996, Jendrol' and Madaras [17] proved $\varphi_M(S_4) = 10$ and $\varphi_T(C_4) = \varphi_T(C_5) = 10$. R. Soták (personal communication, see the surveys of Jendrol' and Voss [19, p.15], [20]) proved $\varphi_P(C_4) = 11$ and $\varphi_P(C_5) = 10$.

In 1999, Jendrol' et al. [18] obtained the following bounds: $10 \leq \varphi_T(C_6) \leq 11$, $15 \leq \varphi_T(C_7) \leq 17$, $15 \leq \varphi_T(C_8) \leq 29$, $19 \leq \varphi_T(C_9) \leq 41$, and $\varphi_T(C_p) = \infty$ whenever $p \geq 11$. Madaras and Soták [24] proved $20 \leq \varphi_T(C_{10}) \leq 415$.

For the broader class $\mathbf{P_5}$, it was known that $10 \leq \varphi_P(C_6) \leq 107$ due to Mohar et al. [25] (in fact, this bound is proved in [25] for all 3-polytopes with $\delta \geq 4$ in which no 4-vertex is adjacent to a 4-vertex). Recently, Borodin et al. [12] proved $\varphi_P(C_6) = \varphi_T(C_6) = 11$.

For C_7, besides the above mentioned result $15 \leq \varphi_T(C_7) \leq 17$, it was known that $\varphi_P(C_7) \leq 359$ (Madaras et al. [23]). Recently, Borodin and Ivanova [8] proved $\varphi_P(C_7) = \varphi_T(C_7) = 15$, which answers a question raised in Jendrol' et al. [18].

The purpose of this paper is to prove the following analogue of Theorem 1.

Theorem 2. *For the class of* 3*-polytopes with minimum degree* 5, $w_P(C_4) = 26$ *and* $w_P(C_5) = 30$.

As an easy corollary, we obtain the above-mentioned unpublished result by R. Soták (for one direction, it suffices to take a C_l with $4 \leq l \leq 5$ of smallest weight and subtract $l - 1$ smallest degrees of its vertices; the other direction follows from the examples in Section 2).

Corollary 3. *For the class of* 3*-polytopes with minimum degree* 5, $\varphi_P(C_4) = 11$ *and* $\varphi_P(C_5) = 10$.

In fact, instead of Theorem 2 we prove the following stronger statement, which extends Theorem 1.

Theorem 4. *Every* 3*-polytope with* $\delta = 5$ *has*

(i) *a* 4*-cycle of weight at most* 26,
(ii) *a* 5*-cycle of weight at most* 30,
(iii) *either a* 4*-cycle of weight at most* 25 *or a facial* 5*-cycle of weight* 25, *where all bounds* 26, 30 *and* 25 *are tight.*

In particular, Theorem 4(i +iii) says that $w_P(C_4)$ can reach its maximum of 26 only in the presence of a facial 5-cycle with weight 25, which is a 5-face completely surrounded by 5-vertices (as in Fig. 1). Theorem 4 refines Corollary 3 as follows.

Corollary 5. *Every* 3*-polytope with* $\delta = 5$ *has*

(i) *a* 4*-cycle of height at most* 11,
(ii) *a* 5*-cycle of height at most* 10,
(iii) *either a* 4*-cycle of height at most* 10 *or a facial* 5*-cycle of height* 5, *where all bounds* 11, 10 *and* 5 *are tight.*

At the second part of the proof of Theorem 4 we use some ideas from Borodin [1] and Borodin–Woodall [9].

Other related structural results on 3-polytopes, some of which have application to coloring, can be found in the already mentioned papers and in [2–8,10–14,24].

2. Proving the tightness of Theorem 4

The bounds in Theorem 4 and Corollary 5 are all precise, as the following examples show. Truncate all vertices of the dodecahedron and cap each 10-face by putting a new vertex inside it and joining it to all the boundary vertices. We have obtained a triangulation with $\delta = 5$ in which $w_T(C_4) = 25$, $w_T(C_5) = 30$, and $\varphi_T(C_4) = \varphi_T(C_5) = 10$.

We now construct a 3-polytope with $w_P(C_4) = 26$ (see Fig. 1). First, we replace each face of the icosahedron as shown in Fig. 1(a). The resulting dual "blue" graph G_1 is a cubic graph with only 5- and 6-faces such that the distance between 5-faces is at least two.

Then, with each 5-face of G_1, we perform the operation depicted in Fig. 1(b) to obtain a graph G_2 with only 3-faces and (very few) 5-faces, in which every vertex is of degree 5, 11, or 12. In particular, we truncate all vertices of G_1 not incident with 5-faces. It is easy to check that each 4-cycle of G_2 goes through an 11^+-vertex, and that $w_P(C_4) = 26$, $\varphi_P(C_4) = 11$, and every facial 5-cycle has weight 25 and hence consists of 5-vertices.