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Neighbor sum distinguishing index of planar graphs

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ABSTRACT

A proper $[k]$-edge coloring of a graph G is a proper edge coloring of G using colors from $[k] = \{1, 2, \ldots, k\}$. A neighbor sum distinguishing $[k]$-edge coloring of G is a proper $[k]$-edge coloring of G such that for each edge $uv \in E(G)$, the sum of colors taken on the edges incident to u is different from the sum of colors taken on the edges incident to v. By $\text{nsdi}(G)$, we denote the smallest value k in such a coloring of G. It was conjectured by Flandrin et al. that if G is a connected graph without isolated edges and $G \neq C_5$, then $\text{nsdi}(G) \leq \Delta(G)+2$. In this paper, we show that if G is a planar graph without isolated edges, then $\text{nsdi}(G) \leq \max\{\Delta(G) + 10, 25\}$, which improves the previous bound ($\max\{2\Delta(G) + 1, 25\}$) due to Dong and Wang.

1. Introduction

The terminology and notation used but undefined in this paper can be found in [3]. Let $G = (V, E)$ be a simple, undirected graph. Let C be a set of colors where $C = [k] = \{1, 2, \ldots, k\}$ and let $\phi : E(G) \to C$ be a proper $[k]$-edge coloring of G. By $m_\phi(v)$ $(C_\phi(v))$, we denote the sum (set) of colors taken on the edges incident to v, i.e. $m_\phi(v) = \sum_{u \in N(v)} \phi(uv)$ $(C_\phi(v) = \{\phi(uv) \mid u \in N(v)\})$. If the coloring ϕ satisfies that $C_\phi(u) \neq C_\phi(v)$ for each edge $uv \in E(G)$, then we call such coloring a *neighbor distinguishing* $[k]$*-edge coloring* of G. We use $\text{ndi}(G)$ to denote the smallest value k such that G has a neighbor distinguishing $[k]$-edge coloring of G and we call it the *neighbor distinguishing index* of G. Sometimes, a neighbor distinguishing edge coloring is named an *adjacent vertex distinguishing edge coloring* [18,19]. If the coloring ϕ satisfies that $m_\phi(v) \neq m_\phi(u)$ for each edge $uv \in E(G)$, then we call such coloring a *neighbor sum distinguishing* $[k]$*-edge coloring* of G. By $\text{nsdi}(G)$, we denote the smallest value k such that G has a neighbor sum distinguishing $[k]$-edge coloring of G and we call it the *neighbor sum distinguishing index* of G.

It is known that to have a neighbor distinguishing or a neighbor sum distinguishing coloring, G cannot have an isolated edge (we call such graphs normal). If a normal graph G has connected components $G_1, \ldots, G_k$, then $\text{ndi}(G) = \max\{\text{ndi}(G_i) \mid i = 1, \ldots, k\}$ and $\text{nsdi}(G) = \max\{\text{nsdi}(G_i) \mid i = 1, \ldots, k\}$. Therefore, when analyzing the neighbor distinguishing index or the neighbor sum distinguishing index, we can restrict our attention to connected normal graphs. Apparently, for any normal graph G, $\Delta(G) \leq \chi'(G) \leq \text{ndi}(G) \leq \text{nsdi}(G)$, where $\chi'(G)$ is the chromatic index of G.

For neighbor distinguishing colorings, we have the following conjecture due to Zhang et al. [23].

Conjecture 1 (*[23]*). *If G is a connected normal graph with at least* 6 *vertices, then* $\text{ndi}(G) \leq \Delta(G) + 2$.

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Akbari et al. [1] proved that $\text{ndi}(G) \le 3\Delta(G)$ for any normal graph G. Hatami [10] has shown that if G is normal and $\Delta(G) > 10^{20}$, then $\text{ndi}(G) \le \Delta(G) + 300$. For more references, see [2,4,7,18,19,11].

Recently, colorings and labelings related to sums of the colors have received much attention. The family of such problems includes e.g. vertex-coloring $[k]$-edge-weightings [13], total weight choosability [21,17], magic and antimagic labelings [12,22] and the irregularity strength [14,15]. As for neighbor sum distinguishing edge colorings, Flandrin et al. [8] completely determined the neighbor sum distinguishing indices for paths, cycles, trees, complete graphs and complete bipartite graphs. Based on these examples, they proposed the following conjecture.

Conjecture 2 ([8]). *If G is a connected normal graph and $G \ne C_5$, then* $\text{nsdi}(G) \le \Delta(G) + 2$.

In the same paper, Flandrin et al. [8] gave an upper bound: $\lceil \frac{7\Delta(G)-4}{2} \rceil$. In [20], Wang and Yan improved it to $\lceil \frac{10\Delta(G)+2}{3} \rceil$. In [16], Przybyło proved that $\text{nsdi}(G) \le 2\Delta(G) + \text{col}(G) - 1$, where $\text{col}(G)$ is the coloring number of G. Dong et al. [6] studied neighbor sum distinguishing colorings of sparse graphs and proved that if G is a normal graph with maximum average degree at most $\frac{5}{2}$ and $\Delta(G) \ge 5$, then $\text{nsdi}(G) \le \Delta(G) + 1$. Dong and Wang [5] also considered the neighbor sum distinguishing colorings of planar graphs and proved the following result.

Theorem 1.1 ([5]). *If G is a connected normal planar graph, then* $\text{nsdi}(G) \le \max\{2\Delta(G) + 1, 25\}$.

In this paper, we improve the result above and obtain the following result.

Theorem 1.2. *If G is a connected normal planar graph, then* $\text{nsdi}(G) \le \max\{\Delta(G) + 10, 25\}$.

2. Preliminaries

First we will introduce some notations. Let G be a graph. For a vertex $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v and $d(v) = |N(v)|$ denote the degree of v. A vertex of degree k is called *k-vertex*. We write *k^+-vertex* for a vertex of degree at least k, and *k^--vertex* for that of degree at most k. Let $N_{k^-}(v) = \{x \in N(v) \mid d(x) \le k\}$ and $n_{k^-}(v) = |N_{k^-}(v)|$. Similarly, $N_{k^+}(v) = \{x \in N(v) \mid d(x) \ge k\}$ and $n_{k^+}(v) = |N_{k^+}(v)|$.

Next we introduce a structural lemma about planar graphs, which was used in [9].

Lemma 2.1 ([9]). *Let G be a planar graph. Then there exists a vertex v in G with exactly $d(v) = t$ neighbors $v_1, v_2, \ldots, v_t$ where $d(v_1) \le d(v_2) \le \cdots \le d(v_t)$ such that at least one of the following is true:*

(A) $t \le 2$,
(B) $t = 3$ *and* $d(v_1) \le 11$,
(C) $t = 4$ *and* $d(v_1) \le 7,\ d(v_2) \le 9$,
(D) $t = 5$ *and* $d(v_1) \le 6,\ d(v_2) \le 7$.

Finally, we give a simple lemma, which will also be used in our proof.

Lemma 2.2 ([8]). *Let z be an integer. For any two sets of integers X, Y, each of size at least 2, there exist (at least) $|X| + |Y| - 3$ pairs $(x_i, y_i) \in X \times Y$ with $x_i \ne y_i,\ i = 1, 2, \ldots, |X| + |Y| - 3$, such that all the sums $x_i + y_i$ are pairwise distinct and among them there are at most two pairs satisfying $x_i - y_i = z$.*

This lemma clearly holds. Indeed, it is sufficient to consider e.g. the pairs from the set

$$(\{x\} \times (Y \setminus \{x\})) \cup ((X \setminus (\{x\} \cup \{y\})) \times \{y\}),$$

where $x = \min X$ and $y = \max Y$.

3. Proof of Theorem 1.2

We prove the theorem by contradiction. Suppose that G is a minimal counterexample with respect to the number of edges. For simplicity, let $\Delta = \Delta(G)$ and $k = \max\{\Delta(G) + 10, 25\}$. Then $k \ge 25$. In the following, we will often delete two adjacent edges, say vv_1, vv_2 to get a subgraph H of G. If H has an isolated edge $e = wp$, then there must be an edge wp in G such that $d_G(w) = 3,\ d_G(p) = 1$ or $d_G(w) = d_G(p) = 2$ or $d_G(w) = 2,\ d_G(p) = 1$. Then $G - wp$ has a neighbor sum distinguishing $[k]$-edge coloring ϕ by the minimality of G. We can easily extend ϕ to the graph G, which is a contradiction. So in the following, we assume that the subgraph H obtained by deleting two adjacent edges from G has no isolated edges.

Claim 3.1. *Let $v \in V(G)$ and v_1, v_2 be the neighbors of v in G. If $d(v_1) \le \frac{k+1-d(v)}{2}$ and $d(v_2) \le \frac{k+1-d(v)}{2}$, then $d(v) \ge \frac{2k-2d(v_1)-2d(v_2)+5}{3}$.*

Proof. Let $H_1 = G - vv_1 - vv_2$. By the minimality of G, H_1 has a neighbor sum distinguishing $[k]$-edge coloring ϕ.

First suppose that v_1 is not adjacent to v_2. For vv_1, we surely cannot use the colors of its (already colored) at most $d(v_1)-1+d(v)-2$ incident edges. Next, the colors in $\{m_\phi(v_2)-m_\phi(v)\} \cup \{m_\phi(u)-m_\phi(v_1) \mid uv_1 \in E(H_1)\}$ are also forbidden. Then we have at least $k-2(d(v_1)-1)-(d(v)-2)-1 \ge k-2d(v_1)-d(v)+3 \ge 2$ safe colors for vv_1. Similarly, we have