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Combinatorial games modeling seki in GO

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ABSTRACT

The game SEKI is played on an $(m \times n)$-matrix A with non-negative integer entries. Two players R (for rows) and C (for columns) alternately reduce a positive entry of A by 1 or pass. If they pass successively, the game is a draw. Otherwise, the game ends when a row or column contains only zeros, in which case R or C wins, respectively. If a zero row and column appear simultaneously, then the player who made the last move is the winner. We will also study another version of the game, called D-SEKI, in which the above case is defined as a draw.

An integer non-negative matrix A is a *seki* or *d-seki* if the corresponding game results in a draw, regardless of whether R or C begins. Of particular interest are the matrices in which each player loses after every option except pass. Such a matrix is called a *complete seki* or a *complete d-seki*. For example, each matrix with entries in $\{0, 1\}$ that has the same sum (at least 2) in each row and column is a complete d-seki, and each such matrix with entries in $\{0, 1, 2\}$ is a complete seki. The game SEKI is closely related to the seki (shared life) positions in the classical game of GO.

1. Introduction

The games SEKI and D-SEKI. The game SEKI was introduced by the first two authors in 1981 in the manuscript [5].

Let $A : I \times J \to \mathbb{Z}_+$ be a non-negative integer $(m \times n)$-matrix having a positive entry in each row and column. The game *SEKI*(A) is defined as follows. Two players R and C alternate turns and it is specified who begins; this player is called the *first*, while the opponent is the *second*. On their turn, the players can either reduce any strictly positive entry of A by 1 (an *active move*) or pass. The game ends in a draw when two players pass successively. A row or column of A is *zero* if all its entries equal 0. Player R wins if a zero row appears before any zero column and player C wins when a zero column appears before any zero row. After some move, a zero row and column may appear simultaneously. In this case, the player who made this last move is claimed the winner. We will also study another version of the game, D-SEKI(A), in which the above case is defined as a draw. Frequently, when only one matrix is under consideration, we will omit the argument A and shorten SEKI(A) and D-SEKI(A) to just SEKI or D-SEKI, respectively. Thus, SEKI results in a draw only after two consecutive passes, while D-SEKI ends in a draw in this situation and also when a zero row and column appear simultaneously.

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Two matrices are isomorphic if one can be obtained from the other by permutations of its rows and columns.

A matrix A is a *seki* or *d-seki* if "perfect play" of both players in the games SEKI(A) or D-SEKI(A), respectively, results in a draw, regardless of whether R or C begins. The concept of perfect play is defined precisely as follows.

Solving SEKI and D-SEKI by backward induction. It is well known that any finite acyclic game can be standardly solved by backward induction [8,9,4], but SEKI and D-SEKI are not acyclic, since any pass is a loop. Nevertheless, some slightly modified backward induction is applicable to both of these games.

For each matrix A we have to solve the games SEKI(A) and D-SEKI(A) for two cases: when R or C begins. Thus, we assign to A one of the nine pairs $(R(A), C(A))$, where $R(A)$ and $C(A)$ take values in $\{W, L, D\}$ (which stand for "win", "lose", and "draw") and show the result for the first player, in other words, for the case when, R and C begin, respectively. For example, $(R(A), C(A)) = (W, W)$ means that the first player wins whether it is R or C, while (W, L) means that R wins whether or not (s)he begins.

We do not consider the matrices that contain more than one zero row or more than one zero column. To initialize the procedure, we assign

- (i) (W, L) to any matrix A that contains exactly one zero row but no zero column;
- (i′) (L, W) to any matrix A that contains exactly one zero column but no zero row;
- (ii) if A contains both one zero row i and one zero column j, then we set $(R(A), C(A)) = (D, D)$ in D-SEKI and $(R(A), C(A)) = (L, L)$ in SEKI.

The assignments (i) and (i′) are obvious: R wins in case (i) and C wins in case (i′), as the first or second player, but (ii) requires some comments. In this case the last move is uniquely defined: the entry $A(i, j)$ is reduced from 1 to 0. After this, the game is over and, according to the rules of SEKI, the player who made this last move wins; hence, we assign (L, L) to SEKI(A). In contrast, D-SEKI is a draw; hence, we assign (D, D) to D-SEKI(A).

After this initialization, the values $R(A)$ and $C(A)$ are defined recursively for both games, SEKI(A) and D-SEKI(A), as follows. Let $N(A)$ denote the set of matrices that can be reached from A by one move. If R begins, then

- $R(A) = W$ whenever $C(A') = L$ for some $A' \in N(A)$;
- $R(A) = D$ if $C(A') = L$ for no $A' \in N(A)$, but $C(A') = D$ for some $A' \in N(A)$.

In other words, R begins and wins (respectively, makes a draw) if (s)he has a move to a position in which C, as the first player, loses (respectively, makes a draw). One case remains:

- [R′] $C(A') = W$ for all $A' \in N(A)$, that is, C wins after any active move by R.

This case is slightly more difficult, since R can still pass, thus, making C the first player. It cannot be a winning move for R, because C can also pass making a draw. Hence, either $R(A) = L$ or $R(A) = D$ in case [R′]. The answer depends on $C(A)$. Let us temporarily set $R(A) = X$ and consider $C(A)$. We define it by symmetry (transposing A) in all cases but

- [C′] $R(A') = W$ for all $A' \in N(A)$, that is, R wins after any active move of C.

In this case, either $C(A) = L$ or $C(A) = D$ and the answer depends on $R(A)$. Let us temporarily set $C(A) = Y$. It looks like a vicious circle, yet, it can be easily broken.

Semi-complete and complete seki. If $R(A) = X$, in case [R′], we redefine it as follows:

- $R(A) = L$ when $C(A) = W$, that is, if C wins after pass of R.
- $R(A) = D$ when $C(A) = D$, that is, if R must pass but C has an active move that still results in a draw. In this case A is called a *semi-complete seki* or *d-seki*.
- $R(A) = D$ when $C(A) = Y$; in other words, both players, R and C, must pass. In this case A is a *complete seki* or *d-seki*.

We define $C(A)$ in case [C′] by symmetry. Notice that A is a complete seki if and only if

- $R(A) = X$ and $C(A) = Y$; in other words both players, R and C, must pass.

Each game, SEKI or D-SEKI, is over after two consecutive passes. Obviously, the optimal result would not change if we allow any number $k \geq 2$ or even an infinite sequence of passes.

The computer code for backward induction. To study the games SEKI and D-SEKI, we generate successively all matrices, up to permutations of their rows and columns, increasing the sum of all entries one by one. Isomorphism-free exhaustive generation (without checking the isomorphism) is a difficult computational problem in general [10]; in particular, no simple efficient procedure for the required matrix generation is known. Yet, for our purposes, we allow some, but not too frequent, repetitions. In this paper we use many examples of non-trivial seki and d-seki computed by this method. The corresponding computer codes were written for SEKI by Konrad Borys and Gabor Rudolf in 2005 and then a more powerful code, working for D-SEKI as well, was written by Diogo Andrade in 2006.

The importance of being first. In both games, SEKI and D-SEKI, for any player, R or C, to be the first is never worse than to be the second. More precisely, if the second player wins or makes a draw in SEKI or D-SEKI, then (s)he can do the same, as the first player. Indeed, it is enough just pass and then apply the same optimal strategy.