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Acyclic vertex coloring of graphs of maximum degree six

Yancai Zhao [a], Lianying Miao [b,*], Shiyou Pang [b], Wenyao Song [b]

[a] Wuxi City College of Vocational Technology, Wuxi, Jiangsu, 221116, PR China

[b] School of Science, China University of Mining and Technology, Xuzhou, Jiangsu, 221008, PR China

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ABSTRACT

In this paper, we prove that every graph with maximum degree six is acyclically 10-colorable, thus improving the main result of Hervé Hocquard (2011).

1. Introduction

A proper vertex coloring of a graph $G = (V, E)$ is an assignment of colors to the vertices of G such that two adjacent vertices do not use the same color. A proper vertex coloring of a graph G is acyclic if G contains no bicolored cycles; in other words, the graph induced by every two color classes is a forest. The acyclic chromatic number of G, denoted by $\chi_a(G)$, is the smallest integer k such that G is acyclically k-colorable. Acyclic colorings were introduced by Grünbaum [10]. The following are some results about acyclic colorings of graphs.

Theorem 1.1 ([10]). *Every planar graph is acyclically* 9*-colorable.*

Theorem 1.2 ([4]). *Every planar graph is acyclically* 5*-colorable.*

This bound is tight since there exist 4-regular planar graphs [10] which are not acyclically 4-colorable.

Theorem 1.3 ([2]). *Every graph with maximum degree* Δ *can be acyclically colored using* $O(\Delta(G)^{4/3})$ *colors.*

Theorem 1.4 ([1]). *Every graph with maximum degree* Δ *can be acyclically colored using* $\Delta(\Delta - 1) + 2$ *colors.*

For graphs with maximum degree six, there are the following results.

Theorem 1.5 ([18]). *Every graph of maximum degree* 6 *can be acyclically colored with* 12 *colors.*

Theorem 1.6 ([11]). *Every graph of maximum degree* 6 *can be acyclically colored with* 11 *colors.*

Other results about the acyclic coloring of graphs can be seen in [1,5,8,6,7,9–16,19]. Here we improve Theorem 1.6 by proving that.

Theorem 1.7. *Every graph with maximum degree six is acyclically* 10*-colorable.*

* Corresponding author.
E-mail addresses: zhaoyc69@126.com (Y. Zhao), miaolianying@cumt.edu.cn (L. Miao).

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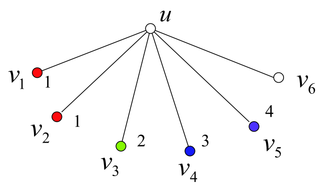

Fig. 1. An illustration of $N_C(u)$, $n_C(u)$, $C_\varphi(u)$ and $c_\varphi(u)$.

This theorem also answers the second question posed by Hervé Hocquard [11].

We now introduce the notations (some of them are first given in [11]) and use the standard graph theory terminology [17] not defined here.

Let $G = (V(G), E(G))$, and $v \in V(G)$. We use $N(v)$ and $d(v)$ to denote the set of the neighbors and the degree of v in G respectively.

A partial acyclic coloring φ of G is an assignment of colors to a subset U of $V(G)$ such that φ is an acyclic coloring of $G[U]$. Let φ be a partial acyclic coloring of G with the color set C and the colored subset $U \subseteq V(G)$ and let v be an uncolored vertex of G. We say that a color $c \in C$ is available for v if no neighbor of v is colored c. We say that a color $c \in C$ is feasible for v if it is available for v and coloring v with c results in a partial acyclic coloring of G. We say that a color $c \in C$ is no-feasible for v if it is available for v and coloring v with c results in bicolored cycles in G. Let F_v and NF_v denote the set of feasible and no-feasible colors for v. For a vertex $u \in V(G)$ (colored or uncolored), we denote the set and the number of colored neighbors of u by $N_C(u) = N(u) \cap U$ and $n_C(u) = |N_C(u)|$ respectively. We denote by $C_\varphi(u)$ the set of colors used by vertices in $N_C(u)$ and $c_\varphi(u) = |C_\varphi(u)|$. For example, in Fig. 1, $N_C(u) = \{v_1, v_2, v_3, v_4, v_5\}$, $n_C(u) = 5$, $C_\varphi(u) = \{1, 2, 3, 4\}$ and $c_\varphi(u) = 4$.

Finally, we denote by $\Delta(G)$, the maximum degree of a graph G. We assume that the graphs in this paper are connected. Let $C = \{1, 2, \ldots, 10\}$.

2. Main result

It is known that [3, P34] every graph of maximum degree at most Δ is an induced subgraph of a Δ-regular graph, and it is sufficient to consider 6-regular connected graphs in this paper.

The following definition is first given in [11].

Let G be a Δ-regular connected graph. A good spanning tree of G is a spanning tree T such that T contains a vertex adjacent to $\Delta - 1$ leaves.

Lemma 1 ([11]). *Every regular connected graph admits a good spanning tree.*

Remark 1. The idea of the proof of Theorem 1.7 is mainly from [11]. We make more careful analysis and use one new technique of constructing bipartite graphs to reduce the number of colors needed to 10.

Proof of Theorem 1.7. Let G be a 6-regular connected graph.

Let T be a good spanning tree of G. Let x_n be a vertex adjacent to five leaves x_1, x_2, x_3, x_4, x_5 in T. We order the vertices of G from x_1 to x_n according to a post-order walk of T. First, we color x_1, x_2, x_3, x_4, x_5 with five distinct colors. Then we will successively color $x_6, x_7, \ldots, x_{n-1}$ while the colors of x_1, x_2, x_3, x_4, x_5 will never be changed. Finally, we color x_n.

Suppose that we have colored $x_1, x_2, \ldots, x_{i-1}$ $(6 \le i \le n-1)$. Let φ be an acyclic 10-coloring of $G_{i-1} = G[x_1, x_2, \ldots, x_{i-1}]$. Now we color $x_i = u$. Since u is adjacent to at least one of $x_{i+1}, \ldots, x_n$, we have $n_C(u) \le 5$. W.l.o.g. assume that $n_C(u) = 5$. Let $N_C(u) = \{v_1, v_2, v_3, v_4, v_5\}$. For $1 \le i, j, k \le 5$, let $N(v_i)\backslash\{u\} = \{v_i^j | 1 \le j \le 5\}$ and $N(v_i^j)\backslash\{v_i\} = \{v_i^{j,k} | 1 \le k \le 5\}$. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$. (See Fig. 2.)

Since $u \ne x_n$, we have the following claim.

Claim 1. *If $v \in N(u)$ and $n_C(v) = 5$, then $v \notin A$. If $v \notin N(u)$ and $n_C(v) = 6$, then $v \notin A$.*

Construction. We construct a bipartite graph H with the bipartition (X, Y) such that $X = \{x | x \in N_C(u)$ and there is a vertex $x' \in N_C(u)$ such that $\varphi(x) = \varphi(x')$ and $x \ne x'\}$ and $Y = NF_u$. For any $x \in X$ and $y \in Y$, x is adjacent to y in H iff assigning u the color y will result in a bicolored cycle passing through u and x. It is easy to see that $d_H(y) \ge 2$ for any $y \in Y$ if $X \ne \phi$ and $Y \ne \phi$.

Now we consider the following five cases.

Case 1. $c_\varphi(u) = 5$. Then there remain five colors for u.

Case 2. $c_\varphi(u) = 4$. W.l.o.g. assume that $\varphi(v_1) = \varphi(v_2) = 1$, $\varphi(v_3) = 2$, $\varphi(v_4) = 3$, $\varphi(v_5) = 4$. We can color u with a color in $C\backslash\{C_\varphi(u) \cup [C_\varphi(v_1) \cap C_\varphi(v_2)]\}$ (which is not ϕ).